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ALGEBRAIC INDEPENDENCE OF VALUES OF EXPONENTIAL AND ELLIPTIC FUNCTIONS

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Dedicated to Professor R.P. Bambah
on the occasion of his 60th birthday

1. Introduction. Let d_1, d_2, m be non-negative integers, with $m > 0$, let $x_1, \ldots, x_{d_1}$ (resp. $y_1, \ldots, y_m$) be complex numbers which are $\mathbf{Q}$-linearly independent, and let $u_1, \ldots, u_{d_2}$ be complex numbers. We consider a Weierstrass elliptic function $\wp$ with invariants g_2, g_3:

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3.$$

We denote by O the ring of endomorphisms of the elliptic curve associated with $\wp$: hence $O = \mathbf{Z}$ if $\wp$ has no complex multiplication, while O is an order of an imaginary quadratic field otherwise. We assume that $u_1, \ldots, u_{d_2}$ are linearly independent over O.

We denote by K_1 the field generated over $\mathbf{Q}(g_2, g_3)$ by the numbers $\exp(x_i y_j)$, $(1 \leqslant i \leqslant d_1,\ 1 \leqslant j \leqslant m)$, together with the numbers $\wp(u_k y_j)$, $(1 \leqslant k \leqslant d_2,\ 1 \leqslant j \leqslant m)$ for which $u_k y_j$ is not a pole of $\wp$.

Next we define

$$K_2 = K_1(y_1, \ldots, y_m),$$
$$K_3 = K_1(x_1, \ldots, x_{d_1}, u_1, \ldots, u_{d_2}),$$

and

$$K_4 = K_1(y_1, \ldots, y_m, x_1, \ldots, x_{d_1}, u_1, \ldots, u_{d_2}).$$

Our main purpose is to give lower bounds for the transcendence degree t_i of K_i over $\mathbf{Q}$, for $i = 1, 2, 3, 4$.

We first give a short historical survey of this problem (§2). Next we state our main result (§3), which is a consequence of a general theorem dealing with algebraic groups (§4). The two next sections are devoted to proofs, and we conclude by announcing some generalizations (§7).

2. Historical survey

(a) Results of transcendence. We get a transcendence result by asserting $t_i > 0$ for some $i = 1, 2, 3, 4$.

Let us begin by the pure exponential case, i.e. $d_2 = 0$. The Hermite-Lindemann's theorem on the transcendence of e^α (for non-zero algebraic α) is equivalent to the assertion $t_4 > 0$ for $d_1 > 0$. Next the Gel'fond-Schneider's theorem on the transcendence of α^β (for algebraic α and β with $\alpha \neq 0$, $\log \alpha \neq 0$ and $\beta \notin \mathbf{Q}$) is equivalent to either one of the following assertions:

$$t_2 > 0 \text{ for } d_1 \geqslant 1 \text{ and } m \geqslant 2$$

$$t_3 > 0 \text{ for } d_1 \geqslant 2.$$

Finally the so-called six-exponentials theorem, due to Siegel, Lang and Ramachandra reads:

$$t_1 > 0 \text{ if } d_1 m > m + d_1.$$

Next we consider the pure elliptic case, i.e. $d_1 = 0$. Schneider's theorem on the transcendence of $\wp(\alpha)$ (for non-zero algebraic α, and for algebraic g_2, g_3) can be written:

$$t_4 > 0 \text{ if } d_2 > 0.$$

The assertions

$$t_2 > 0 \text{ for } d_2 \geqslant 1 \text{ and } m \geqslant 3$$

and

$$t_3 > 0 \text{ for } d_2 \geqslant 2$$

are also due to Schneider, while Lang and Ramachandra proved

$$t_1 > 0 \text{ if } d_2 m > m + 2d_2.$$

Finally, in the general case ($d_1 \geqslant 0$, $d_2 \geqslant 0$), the statement

$$t_3 > 0 \text{ for } d_1 \geqslant 1 \text{ and } d_2 \geqslant 1$$

is due to Schneider, the inequality

$$t_1 > 0 \text{ for } m(d_1 + d_2) > m + 2(d_1 + d_2)$$

can be deduced from a result of Lang on algebraic groups, while the refinement

$$t_1 > 0 \text{ for } m(d_1 + d_2) > m + d_1 + 2d_2$$

is a consequence of Ramachandra's work on functions satisfying an addition theorem.

In conclusion, the following statement includes results due to Hermite, Lindemann, Schneider, Gel'fond, Lang and Ramachandra (see [G], [L], [R], and [S]).

THEOREM 1.1.

(i) *If* $(d_1 + d_2)m > m + d_1 + 2d_2$, *then* $t_1 \geqslant 1$.

(ii) *If* ($d_1 \geqslant 1$ *and* $m \geqslant 2$), *or* ($d_2 \geqslant 1$ *and* $m \geqslant 3$), *then* $t_2 \geqslant 1$.

(iii) *If* $d_1 + d_2 \geqslant 2$, *then* $t_3 \geqslant 1$.

(iv) *If* $d_1 + d_2 \geqslant 1$, *then* $t_4 \geqslant 1$.

The relevant values for the assumption $m(d_1 + d_2) > m + d_1 + 2d_2$ are as follows:

$m \geqslant$	2	3	3	3	4	4	5
$d_1 \geqslant$	3	2	1	0	1	0	0
$d_2 \geqslant$	0	0	2	4	1	3	2

(b) Small transcendence degree. If we have $t_i \geqslant 0$ for some $i = 1, 2, 3, 4$, then two at least of the elements of K_i are algebraically independent.

In the pure exponential case, the first results were due to Gel'fond [G]. They have been refined by Shmelev, Tijdeman, Brownawell,... (see [W2]) and read now:

if $d_1 m \geqslant 2(m + d_1)$, then $t_1 \geqslant 2$

if $d_1 m \geqslant m + 2d_1$, then $t_2 \geqslant 2$; if $d_1 m \geqslant 2m + d_1$, then $t_3 \geqslant 2$

if $d_1 m > m + d_1$, then $t_4 \geqslant 2$.

(Of course, the second and third results are equivalent).

In the pure elliptic case, the first results were due to Brownawell and Kubota [B-K]. Later using their zero estimate, Masser and Wüstholz [M-W1] proved the expected elliptic analog of the above mentioned result; the rule is to replace d_1 by d_2 in the left hand side of the assumed inequality, and by $2d_2$ in the right hand side.

Finally, if one combines these results with recent works of Tubbs [T] and with theorem 2 of [W4], one concludes:

THEOREM 1.2.

(i) *If* $(d_1 + d_2)m \geqslant 2(m + d_1 + 2d_2)$, *then* $t_1 \geqslant 2$.

(ii) *If* $(d_1 + d_2)m \geqslant m + 2(d_1 + 2d_2)$, *then* $t_2 > 2$.

(iii) *If* $(d_1 + d_2)m \geqslant 2m + d_1 + 2d_2$, *then* $t_3 \geqslant 2$.

(iv) *If* $(d_1 + d_2)m > m + d_1 + 2d_2$, *then* $t_4 \geqslant 2$.

Several refinements, involving extra assumptions (periodicity, assumption that some of the considered numbers are algebraic, torsion points,...) are known (see [C], [T], [W2], [W4]).

There are also further results on "small transcendence degrees", in particular due to Chudnovsky [C] (see also [W2]), which yield $t_i \geqslant 3$ for some i and which are slightly sharper then the results on "large transcendence degrees" we are going to discuss now.

(c) Large transcendence degrees. So far, in order to get lower bounds for t_i in terms of d_1, d_2, m which yield stronger results than $t_i \geqslant 2$, one needs for technical reasons extra assumptions on measures of linear independence of the numbers x_i, y_j and u_k. It is an interesting and non trivial open problem to remove these assumptions.

We assume that for all $\varepsilon > 0$ there exists $H_0 > 0$ such that for all $(h_1, \ldots, h_{d_1}) \in \mathbf{Z}^{d_1}$ with $\max_{1 \le i \le d_1} | h_i | = H > H_0$,

$$| h_1 x_1 + \ldots + h_{d_1} x_{d_1} | > \exp(-H^{\varepsilon}). \tag{2.1}$$

Similarly, we assume that for all $\varepsilon > 0$ there exists $L_0 > 0$ such that for all $(\lambda_1, \ldots, \lambda_m) \in \mathbf{Z}^m$ with $\max_{1 \le j \le m} | \lambda_j | = L > L_0$,

$$| \lambda_1 y_1 + \ldots + \lambda_m y_m | \exp(-L^{\varepsilon}). \tag{2.2}$$

Finally, we assume that for all $\varepsilon > 0$, there exists $M_0 > 0$ such that for all $(m_1, \ldots, m_{d_2}) \in O^{d_2}$ with $\max_{1 \le k \le d_2} | m_k | = M > M_0$,

$$| m_1 u_1 + \ldots + m_{d_2} u_{d_2} | > \exp(-M^{\varepsilon}). \tag{2.3}$$

From now on in this section we shall assume that these assumptions hold.

The first result on large transcendence degrees (apart from the Lindemann-Weierstrass theorem which we mention briefly in §7 below) was provided by Chudnovsky in 1974, in the pure exponential case:

$$2^{t_1} > d_1 m/(m + d_1),$$
$$2^{t_2} > (d_1 + 1)m/(m + d_1) \text{ and } 2^{t_3} > d_1(m + 1)/(m + d_1),$$
$$2^{t_4} \geqslant (d_1 m + m + d_1)/(m + d_1).$$

Further works on this subject are due to Warkentin, Philippon, Reyssat, Endell and Nesterenko (see [W2] II §1). Recently, Philippon succeeded to prove a sharp refinement of these results, by replacing, essentially, 2^t by $t + 1$:

$$t_1 \geqslant (d_1 m - d - m)/(m + d_1),$$
$$t_2 \geqslant d_1(m - 1)/(m + d_1) \text{ and } t_3 \geqslant (d_1 - 1)m/(m + d_1),$$
$$t_4 \geqslant d_1 m/(m + d_1).$$

In the pure elliptic case ($d_1 = 0$), Masser and Wüstholz [M-W2] proved

$$2^{t_1+2}\cdot(t+8) \geqslant md_2/(m+2d_2),$$

under the assumption that g_2 and g_3 are algebraic (they also assumed $O = \mathbf{Z}$, but it is easy to remove this assumption). The refinement

$$2^{t_1} \geqslant md_2/(m+2d_2)$$

is given in [W3], without the assumption that g_2, g_3 are algebraic, and the better result

$$t_1 \geqslant (md_2 - m - 2d_2)/(m+2d_2)$$

(see [W3], cor. 13.3) involves a further technical assumption: let $\tau = \omega_2/\omega_1$ be a quotient of a pair of fundamental periods of $\wp$; we assume that for all $\varepsilon > 0$ there exists $H_0 > 0$ such that for all $H > H_0$ and all imaginary quadratic number β of height $\leqslant H$, with $\tau \neq \beta$,

$$|\tau - \beta| > \exp(-H^{\varepsilon}). \qquad (2.4)$$

Masser [M] (Th. I, p. 1) proved that this assumption (2.4) is automatically satisfied when the modular invariant

$$j = 1728\, g_2^3/(g_2^3 - 27g_3^2)$$

is algebraic (for sharper estimates see [F–P]). It should not be too difficult to remove completely this assumption (2.4).

3. The main result.

THEOREM 3.1. *We assume that the conditions* (2.1), (2.2), (2.3) *and* (2.4) *hold. Then*

$$t_1 \geqslant ((d_1+d_2)\,m - m - d_1 - 2d_2)/(m + d_1 + 2d_2),$$

$$t_2 \geqslant ((d_1+d_2)\,m - d_1 - 2d_2)/(m + d_1 + 2d_2),$$

$$t_3 \geqslant (d_1 + d_2 - 1)\,m/(m + d_1 + 2d_2),$$

$$t_4 \geqslant (d_1 + d_2)\,m/(m + d_1 + 2d_2).$$

It should be pointed out that these lower bounds improve Phillipon's one (which correspond to $d_2 = 0$) only if $m > d_1$ for t_1 and t_4, $m > d_1 + 2$ for t_2, and $m > d_1 - 2$ for t_3.

The lower bound for t_4, say, is a very slight improvement compared with the lower bound for t_1, since one adds only one in the right hand side, while we adjoin $d_1 + d_2 + m$ numbers to K_1 in order to get K_4. Because of this we will give the proof only for the lower bounds of t_1 and t_2 (which involve Schneider's method, while the two others involve Gel'fond's method).

Instead of taking only one $\wp$-function, it is possible to consider several elliptic functions. However the technical assumptions (2.3) and (2.4) have to be modified accordingly (see [W3] Cor. 13.6). Let us give an example. *Let $E_i = \mathbf{C}/\Omega_i$ $(1 \leqslant i \leqslant d)$ be elliptic curves over $\overline{\mathbf{Q}}$ which are pairwise non-isogeneous. For $1 < i \leqslant d$, let $\omega_i \in \Omega_i$, $\omega_i \neq 0$. Next, let $y_1, \ldots, y_m$ be $\mathbf{Q}$-linearly independent numbers. We assume that for all $\varepsilon > 0$ there exists $M_0 > 0$ such that for all $i = 1, \ldots, d$, for all $\omega \in \Omega_i$, and all $(h_1, \ldots, h_m) \in \mathbf{Z}^m$ with $\max_{1 \leqslant j \leqslant m} |h_j| = M \geqslant M_0$, if the number*

$$\xi = \omega - \omega_i \cdot \sum_{j=1}^{m} h_j v_j$$

does not vanish, then

$$|\xi| > exp(-M^{\varepsilon}).$$

Then the transcendence degree of the field generated by $\wp_i(\omega_i y_j)$, $(1 \leqslant i < d$, $i \leqslant j \leqslant m)$ is at least $2(dm - 2d - m + 1)/(m + 2d - 1)$.

It is also possible to consider Weierstrass zeta or sigma functions. Indeed, the general result which can be proved involves algebraic groups. It is also possible to translate such a result in terms of functions satisfying an algebraic addition theorem (see [R]) by means of a result of Weil [We].

4. Algebraic groups. This section is devoted to the statement of a somewhat complicated result, which contains the hard part of the proof of large transcendence degrees, excluding the zero estimate.

Let K be a subfield of $\mathbf{C}$ of transcendence degree t over $\mathbf{Q}$. Let d_0, d_1, d_2 be non-negative integers with $d = d_0 + d_1 + d_2 > 0$. Define

$$G_0 = \mathbf{G}_a^{d_0},\ G_1 = \mathbf{G}_m^{d_1},$$

and let G_2 be a commutative algebraic group of dimension d_2 which is defined over K, and suitably embedded in a projective space $\mathbf{P}_N$ over K (see [W3]). Further define $G = G_0 \times G_1 \times G_2$ in $\mathbf{P}_{d_0} \times \mathbf{P}_{d_1} \times \mathbf{P}_N = \mathbf{P}$.

Let n be a positive integer, and $\varphi: \mathbf{C}^n \to G(\mathbf{C})$ be an analytic homomorphism. We assume that the tangent map $d_0\varphi: \mathbf{C}^n \to T_G(\mathbf{C})$ of φ at the origin is injective, and that $\varphi(\mathbf{C}^n)$ is Zariski dense in $G(\mathbf{C})$. Further, we put $\kappa = \operatorname{rank}_{\mathbf{Z}} \ker \varphi$.

Let $Y = \mathbf{Z}y_1 + \ldots + \mathbf{Z}y_m$ be a finitely generated subgroup of $\mathbf{C}^n$ of rank m over $\mathbf{Z}$, such that $\Gamma = \varphi(Y)$ is contained in $G(K)$. Further let $l = \operatorname{rank}_{\mathbf{Z}} \Gamma$. Furthermore, let $\mu^{\#}$ and ν be positive real numbers such that

$$1 + t\left(1 - \frac{\kappa}{2n}\right) < (d\mu^{\#} + \kappa - d_1 - 2d_2)/n\mu^{\#} \tag{4.1}$$

and

$$t + 1 < \nu < \left(d - \frac{\kappa}{2} + \frac{\kappa - d_1 - 2d_1}{\mu^{\#}}\right)\Big/\left(n - \frac{\kappa}{2}\right). \tag{4.2}$$

We choose a transcendence basis $(\theta_1, \ldots, \theta_t)$ of K over $\mathbf{Q}$, and a primitive element $\theta_{t+1} \in K$ of K over $\mathbf{Q}(\theta_1, \ldots, \theta_t)$, which is integral over $\mathbf{Z}[\theta_1, \ldots, \theta_t]$. Let $B \in \mathbf{Z}[X_1, \ldots, X_{t+1}]$ be such that $B(\theta_1, \ldots, \theta_t, X)$ is the minimal polynomial of θ_{t+1} over $\mathbf{Q}(\theta_1, \ldots, \theta_t)$.

There exist a constant $c > 0$ such that, for each $(\tilde{\theta}_1, \ldots, \tilde{\theta}_t) \in \mathbf{C}^t$ with

$$\max_{1 \leqslant \tau \leqslant t} |\tilde{\theta}_\tau - \theta_\tau| \leqslant c,$$

we can define $\tilde{\theta}_{t+1}$ to be the unique root of $B(\tilde{\theta}_1, \ldots, \tilde{\theta}_t, X)$ which is at minimal distance of θ_{t+1}, we can construct an algebraic group $\tilde{G}$ defined over the field $\tilde{K} = \mathbf{Q}(\tilde{\theta}_1, \ldots, \tilde{\theta}_{t+1})$, and we can also deform the points $\gamma_j = \varphi(y_j)$ of $G(K)$ into points $\tilde{\gamma}_j$ of $\tilde{G}(\tilde{K})$ (see [W3] for details).

Finally, we choose a sufficiently large integer $c_0 > 0$, next an integer S_0 much larger than c_0. For each $S > S_0$, we define D_0, D_1, D_2, Δ as functions of S by

$$D_0 \log S = D_1 S = D_2 S^2 = c_0^{-1} \cdot S^{\mu^{\#}}.$$

THEOREM 4.3. *There exists an unbounded set of real numbers* $S > S_0$ *with the following property. For each such* S, *there exists* $(\tilde{\theta}_1, \ldots, \tilde{\theta}_t) \in \mathbf{C}^t$, *satisfying*

$$\max_{1 \leqslant \tau \leqslant t} |\tilde{\theta}_\tau - \theta_\tau| < \exp(-2S^\nu)$$

such that, if we define $\tilde{\theta}_{t+1}, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_m, \tilde{G}$ *as before, then the set*

$$\tilde{\Gamma}(S) = \{h_1\tilde{\gamma}_1 + \ldots + h_m\tilde{\gamma}_m;\ h_j \in \mathbf{Z},\ 0 \leqslant h_j \leqslant S,\ 1 \leqslant j \leqslant m\}$$

is contained in an algebraic hypersurface of $\mathbf{P}$, *of multidegrees at most* (D_0, D_1, D_2), *which does not contain* $\tilde{G}(\tilde{K})$.

We sketch the proof of theorem 4.3 in §5 below, while in §6 we show how to deduce theorem 3.1 by using a zero estimate due to Philippon.

Notice that theorem 4.3 is a refinement of [W3] Th. 9.1.

5. Proof of Theorem 4.3. The proof is a slight modification of the one given in [W3]. We just mention the differences.

In the first stage, we replace the assumptions (4.1) and (4.2) by the stronger ones

$$1 + t\left(1 - \frac{\kappa}{2n} + \frac{1}{n}\right) < \frac{d}{n} + \frac{\kappa - d_1 - 2d_2}{n\mu^{\#}} \tag{5.1}$$

and

$$t + 1 < \nu < \left(d + 1 - \frac{\kappa}{2} + \frac{\kappa - d_1 - 2d_2}{\mu^{\#}}\right)\Big/\left(n + 1 - \frac{\kappa}{2}\right) \tag{5.2}$$

We apply Proposition 2.1 of [W1] with U and r defined by

$$U^{n+1-\kappa/2} = c_0^{-3n} \cdot \Delta^{d+1-\kappa/2} S^{-d_1-2d_2+\kappa} (\log S)^{-d_0},$$

and

$$r = S(U/c_0\Delta)^{1/2}.$$

This choice is done in such a way that

$$D_2 r^2 \leqslant c_0 U \text{ (with } D_2 = \Delta/S^2)$$

and

$$U^{n+1} \leqslant c_0^{-2} D_0^{d_0} D_1^{d_1} D_2^{d_2} r^\kappa \Delta,$$

while the assumption (5.1) implies

$$\Delta^{t+1} \leqslant c_0^{-2} U.$$

We construct an auxiliary function using the method of [W1] §2. The main two differences with the proof of [W1] Prop. 2.4 are the following ones:

1. The values of the parameters U and r are different.

2. We replace $\overline{\mathbf{Q}}$ by K; hence, instead of using Liouville's inequality ("deuxième pas" in [W1] p. 641], we use Philippon's criterion ([P1]; see also [W3] §2).

This yields the conclusion of theorem 4.3 under the assumptions (5.1) and (5.2).

The last stage is to use the so-called Landau-Philippon's trick: assuming only (4.1) and (4.2), we choose an integer $k \geq 1$ large enough such that

$$1 + t\left(1 - \frac{\kappa}{2n} + \frac{1}{nk}\right) < \frac{d}{n} + \frac{\kappa - d_1 - 2d_2}{n\mu^{\#}}$$

and

$$t + 1 < \nu < \left(d + \frac{1}{k} - \frac{\kappa}{2} + \frac{\kappa - d_1 - 2d_2}{\mu^{\#}}\right) \Big/ \left(n + \frac{1}{k} - \frac{\kappa}{2}\right),$$

and we apply the result we just proved to Y^k, G^k, with $n, \kappa, d_0, d_1, d_2, d$ replaced by $kn, k\kappa, kd_0, kd_1, kd_2, kd$, while t, $\mu^{\#}$, ν are unchanged (compare with [W3] lemma 5.3). It is convenient here to introduce the following notation:

Definition. Let $G = \mathbf{G}_a^{d_0} \times \mathbf{G}_m^{d_1} \times G_2$ be an algebraic group of dimension $d = d_0 + d_1 + d_2$ as before, and $\Gamma = \mathbf{Z}\gamma_1 + \ldots + \mathbf{Z}\gamma_m$ be a finitely generated subgroup of $G(K)$. For each integer $S \geq 1$, we define $\omega^{\#}(\Gamma(S), G)$ as the minimum of the real numbers $\Delta > 0$ such that there exists a multihomogeneous polynomial of multidegrees at most (D_0, D_1, D_2), with

$$D_0 \log S = D_1 S = D_2 S^2 = \Delta$$

which vanishes on $\Gamma(S)$, but not on all G.

Lemma 5.3. *For each integer* $k \geq 1$,

$$\omega^{\#}(\Gamma^k(S), G^k) = \omega^{\#}(\Gamma(S), G).$$

We conclude the proof of theorem 4.3 by using lemma 5.3 for $\tilde{\Gamma}$ and $\tilde{G}$.

6. Proof of Theorem 3.1. The difference now with the proof of [W 3] is that we replace the zero estimate of Masser-Wüstholz (see [M-W 2] Chap. 1, and [W 3] §6) by a refinement due to Philippon [P 2].

DEFINITION. Let $G = \mathbf{G}_a^{d_0} \times \mathbf{G}_m^{d_1} \times G_2$ be an algebraic group of dimension $d = d_0 + d_1 + d_2$ over a subfield K of $\mathbf{C}$, and let Γ be a finitely generated subgroup of $G(K)$. We define

$$\mu^{\#}(\Gamma, G) = \min_H (\lambda + r_1 + 2r_2)/r,$$

where H runs over the algebraic subgroups of G defined over K, with $H \neq G$, $r = \dim G/H$, and r_0, r_1 are the largest integers such that

$$G/H = \mathbf{G}_a^{r_0} \times \mathbf{G}_m^{r_1} \times G',$$

and where $r_2 = r - r_0 - r_1$, and $\lambda = \operatorname{rank}_{\mathbf{Z}} \Gamma/\Gamma \cap H$.

A corollary of Philippon's result in [P 2] is:

$$\omega^{\#}(\Gamma(S), G) \geqslant cS^{\mu^{\#}(\Gamma, G)},$$

where c does not depend on S.

More precisely, under the hypotheses of theorem 4.3 there exists an algebraic subgroup H of $\tilde{G}$, with $\tilde{G}/H = \mathbf{G}_a^{r_0} \times \mathbf{G}_m^{r_1} \times G'$, $\dim G' = r_2$, $r = r_0 + r_1 + r_2$, say, where H is defined by equation of multidegrees at most $(D_0, D_1, c_0 D_2)$, and there exist elemets $h^{(1)}, \ldots, h^{(l-\lambda)}$ of

$$\mathbf{Z}_{\pm}^m(S) = \{(h_1, \ldots, h_m) \in \mathbf{Z}^m, \mid h_j \mid \leqslant S,\ 1 \leqslant j \leqslant m\}$$

which are $\mathbf{Z}$-linearly independent, with

$$0 < \lambda \leqslant l,\ 1 \leqslant r \leqslant d,\ (\lambda + r_1 + 2r_2)/r \lhd \mu^{\#},$$

such that, if $h^{(s)} = (h_1^{(s)}, \ldots, h_m^{(s)})$, then

$$\sum_{j=1}^m h_j^{(s)} \tilde{\gamma}_j \in H(\tilde{K}), \qquad (1 \leqslant s < l - \lambda).$$

Now, for the proof of the two first inequalities in theorem 3.1, we use what we just proved for $G_2 = E^{d_2}$, where E is the elliptic curve associated with $\wp$, and $d_0 = 0$ when we work with t_1, while $d_0 = 1$ for t_2.

Next we follow the proof given in [W 3] §13: one uses a refined version of Kolchin's theorem [M-W 2] (Chap. III) in order to describe explicitely the algebraic subgroups of $\tilde{G} = \mathbf{G}_a^{d_0} \times \mathbf{G}_m^{d_1} \times \tilde{E}^{d_2}$. The assumption (2.4), together with [W 3] lemma 13.10, enables one to write

$$H = H_0 \times H_1 \times \tilde{H}_2,$$

where H_2 is an algebraic subgroup of E^{d_2}. Finally, one uses [W 3] lemma 13.9 (in the simplest case $n = 1$) to get a contradiction with assumption (2.1), (2.2) and (2.3).

7. Further results.

(a) Using the arguments of [W 1] and [W 3], it is not difficult to extend theorem 3.1 to n-variables. One takes x_i, y_j, u_k in $\mathbf{C}^n$, one replaces the products like $x_i y_j$ by the usual scalar product in $\mathbf{C}^n$, and the ranks like $m = \operatorname{rank}_{\mathbf{Z}} Y$ by the Dirichlet exponent $\mu(Y, \mathbf{C}^n)$.

(b) Similar results can be proved in the p-adic case too.

(c) As mentioned earlier, we proved the lower bounds for t_1 and t_2 only, because the results for t_3 and t_4 involve only very small improvements. However, there is one circumstance where the use of derivatives yields strong results, namely the situation of theorems of Lindemann-Weierstrass type. Chudnovsky was the first to notice that Gel'fond's method can be used to prove for instance the algebraic independence of e^{α_1} and e^{α_2}, or of $\wp(\alpha_1)$ and $\wp(\alpha_2)$ in the CM case (where α_1, α_2 are algebraic number which are linearly independent over $\mathbf{Z}$ or O respectively). A survey of recent works in this direction is given in [P 3].

(d) It seems reasonable to expect that, in the situation of §4, the lower bound

$$1 + t \geqslant (d\mu^{\#}(\Gamma, G) + \kappa - d_1 - 2d_2)/n\mu^{\#}(\Gamma, G)$$

holds, without further assumptions. This would yield

$$1 + t > d\mu(Y, \mathbf{C}^n)/(n\mu(Y, \mathbf{C}^n) + d_1 + 2d_2 - \kappa).$$

The next step would be to replace (d_0, d_1, d_2) by (d_a, d_m, g) when G is an extension of an abelian variety of dimension g by $\mathbf{G}_a^{d_a} \times \mathbf{G}_m^{d_m}$.

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