J. Austral. Math. Soc. (Series A) 25 (1978), 466–478

SIMULTANEOUS APPROXIMATION OF NUMBERS CONNECTED WITH THE EXPONENTIAL FUNCTION

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Dedicated to Kurt Mahler on his 75th birthday

(Received 16 January 1978)

Communicated by J. H. Coates

Abstract

We give several results concerning the simultaneous approximation of certain complex numbers. For instance, we give lower bounds for $|\ a-\xi_0\ |+|\ e^a-\xi_1\ |$, where a is any non-zero complex number, and ξ_0, ξ_1 are two algebraic numbers. We also improve the estimate of the so-called Franklin Schneider theorem concerning $|\ b-\xi_0\ |+|\ a-\xi_1\ |+|\ a^b-\xi_2\ |$. We deduce these results from an estimate for linear forms in logarithms.

Subject classification (*Amer. Math. Soc.* (*MOS*) 1970): primary 10 F 10; secondary 10 F 35.

1. Introduction

When α is an algebraic number, we denote by $H(\alpha)$ the height (in the usual sense) of α. In the present paper we derive several consequences of the following result.

THEOREM 1.1. *Let $\alpha_1,\dots,\alpha_n$ be non-zero algebraic numbers, and $\beta_0,\beta_1,\dots,\beta_n$ be algebraic numbers. For $1\leqslant j\leqslant n$, let $\log\alpha_j$ be any determination of the logarithm of α_j. Let D be a positive integer, and $A_0,A_1,\dots,A_n,B,E_1$ be positive real numbers, satisfying*

$$D\geqslant[\mathbf{Q}(\alpha_1,\dots,\alpha_n,\beta_0,\dots,\beta_n):\mathbf{Q}],$$

$$A_j\geqslant\max\{H(\alpha_j),\exp|\log\alpha_j|,e\}\quad(1\leqslant j\leqslant n),$$

$$B\geqslant\max_{0\leqslant j\leqslant n}H(\beta_j),$$

$$A_n\geqslant A_{n-1}\geqslant\dots\geqslant A_1,\quad A_0=e$$

and

$$1 < E_1 \leqslant \min_{1\leqslant j\leqslant n} \{e(\operatorname{Log} A_j)/|\log \alpha_j|\}.$$

If the number

$$\Lambda = \beta_0 + \beta_1 \log \alpha_1 + \ldots + \beta_n \log \alpha_n$$

does not vanish, then

$$|\Lambda| > \exp\{-C_1(n)\, D^{n+2}.(\operatorname{Log} A_1)\ldots(\operatorname{Log} A_n).(\operatorname{Log} B + \operatorname{Log}\operatorname{Log} A_n + \operatorname{Log} E_1)$$
$$.(\operatorname{Log}\operatorname{Log} A_{n-1} + \operatorname{Log} E_1)(\operatorname{Log} E_1)^{-n-1}\},$$

where

$$C_1(1) \leqslant 2^{39}, \quad C_1(2) \leqslant 2^{59} \quad \textit{and} \quad C_1(n) \leqslant 2^{10n+53}.n^{2n}.$$

We will deduce this result from Theorem C of [Wa] in Section 2 below. The connection with a previous result of Baker is the following. In [Ba], Baker sets

$$\Omega = \prod_{j=1}^{n} \operatorname{Log}\max\{H(\alpha_j), 4\}$$

and

$$\Omega' = \prod_{j=1}^{n-1} \operatorname{Log}\max\{H(\alpha_j), 4\},$$

and proves that

$$|\Lambda| > \exp\{-(16nD)^{200n}\,\Omega(\operatorname{Log}(B\Omega))\operatorname{Log}\Omega'\},$$

provided the logarithms are principal valued. This last requirement implies

$$|\log \alpha| \leqslant \pi + \operatorname{Log}(H(\alpha)+1).$$

Therefore, by choosing $E_1 = e$ in Theorem 1.1, we can replace $(16nD)^{200n}$ (in Baker's result) by $2^{12n+53}.n^{2n}.D^{n+2}$. However, several of our applications will involve a large value for E_1, which leads to an improved bound for $|\Lambda|$. For instance, when the numbers $|\log \alpha_j|$ are bounded, we obtain the following result (choosing $\operatorname{Log} A_j = R \operatorname{Log} H_j$, $E_1 = \operatorname{Log} H_1$).

COROLLARY 1.2. *With the notations of Theorem* 1.1, *let* $H_0, H_1, \ldots, H_n, R$ *satisfy*

$$R \geqslant 1 + \max_{1\leqslant j\leqslant n} |\log \alpha_j|,$$

$$H_j \geqslant \max\{H(\alpha_j), e^e\} \quad (1 \leqslant j \leqslant n)$$

and

$$H_0 = H_1 \leqslant \ldots \leqslant H_n,$$

then

$$|\Lambda| > \exp\{-C_2(n,R)\, D^{n+2}(\operatorname{Log} H_1)\ldots(\operatorname{Log} H_n)(\operatorname{Log} B + \operatorname{Log}\operatorname{Log} H_n)$$
$$.(\operatorname{Log}\operatorname{Log} H_{n-1})(\operatorname{Log}\operatorname{Log} H_1)^{-n-1}\},$$

with

$$C_2(n,R) \leqslant C_1(n).R^n.(2+\operatorname{Log} R)^2.$$

In the present paper, we first discuss a problem of K. Mahler on $|e^n-p|$; we then consider the simultaneous approximation of a and e^a; then we deal with Franklin Schneider's theorem [S], and some of its generalizations. Finally, we derive a connection between simultaneous approximations and algebraic independence.

Here, we do not pay a special attention to the degree. Our results will be rather sharp with this respect, but we could improve them by using the refined arguments of [Wa], at the cost of complicating the statements.

This paper has been written at the Australian National University in Canberra (Institute for Advanced Studies).

2. Auxiliary results

We first show how to deduce Theorem 1.1 from Theorem C of [Wa]. Let us define

$$V_j = \operatorname{Log} A_j \quad (1 \leqslant j \leqslant n),$$

$$W = \operatorname{Log} B$$

and

$$E = (DE_1)^{\frac{1}{3}}.$$

In view of the inequalities

$$W + \operatorname{Log} V_n + \operatorname{Log} E + \operatorname{Log} D \leqslant \tfrac{4}{3}(\operatorname{Log} B + \operatorname{Log}\operatorname{Log} A_n + \operatorname{Log} E_1 + \operatorname{Log} D)$$

and

$$\operatorname{Log} V_{n-1} + \operatorname{Log} E + \operatorname{Log} D \leqslant \tfrac{4}{3}(\operatorname{Log}\operatorname{Log} A_{n-1} + \operatorname{Log} E_1 + \operatorname{Log} D)$$

we will obtain $C_1(n) \leqslant 3^{n-1}2^4 C(n) \leqslant 2^{2n+2} C(n)$, (where $C(n)$ is the constant of Theorem C of [Wa]), provided that we prove

$$E \leqslant e^{DV_1}.$$

Since

$$e.2^D.D^2 \log A_1 \leqslant A_1^{3D-1},$$

it is sufficient to prove the following lemma.

LEMMA 2.1. *Let* α *be a non-zero algebraic number of degree at most* D *and height at most* A, *and let* $\log\alpha$ *be a non-zero determination of the logarithm of* α. *Then*

$$|\log\alpha| \geqslant 2^{-D}.(AD)^{-1}.$$

PROOF OF LEMMA 2.1. Since $2^D AD \geqslant 2$, we may assume $|\log\alpha| \leqslant \frac{1}{2} < \mathrm{Log}\, 2$. From the inequality $|e^z - 1| \leqslant |z| e^{|z|}$ for all $z \in \mathbf{C}$ we deduce

$$|\alpha - 1| \leqslant 2|\log\alpha|.$$

Finally, we have (for example by using Lemma 3 of [M–W])

$$|\alpha - 1| \geqslant 2^{1-D}(AD)^{-1}.$$

This completes the proof of Lemma 2.1.

The following simple lemma is proved in [Wa], Lemma 2.4.

LEMMA 2.2. *Let v and w be two complex numbers satisfying*

$$|w - e^v| \leqslant \tfrac{1}{3}|e^v|.$$

Then there exists a determination of the logarithm of w such that

$$|w - e^v| \geqslant \tfrac{2}{3}|e^v|\, |\log w - v|.$$

We now prove several auxiliary lemmas which will be used in Section 6. We use the notations of [Wa].

LEMMA 2.3. *Let $P \in \mathbf{Z}[X_0, \ldots, X_m]$ be a polynomial of degree at most N_j with respect to X_j $(0 \leqslant j \leqslant m)$, let $\alpha_1, \ldots, \alpha_m$ be algebraic numbers generating a field K of degree at most D, such that the polynomial $P(X_0, \alpha_1, \ldots, \alpha_m) \in \mathbf{C}[X_0]$ does not vanish identically. Let t be a complex number.*

There exist a positive integer k, and an algebraic number γ of degree at most DN_0/k, such that

$$M(\gamma)^k \leqslant L(P)^D . \exp\left\{\sum_{j=1}^{m} N_j h(\alpha_j)\right\}$$

and

$$|\gamma - t|^k \leqslant |P(t, \alpha_1, \ldots, \alpha_m)|\, 2^{4D^2N_0^2} . L(P)^{D^2N_0 + D - 1} . \max\{1, |t|\}^{N_0(D-1)} . \exp\left\{(1 + DN_0)\sum_{j=1}^{m} N_j h(\alpha_j)\right\}.$$

PROOF OF LEMMA 2.3. For $1 \leqslant j \leqslant m$, let a_j be the leading coefficient of the minimal polynomial of α_j, with, say, $a_j > 0$, and let d_j be the degree of α_j. We denote by $\{\sigma\}$ the set of the embeddings of K into $\mathbf{C}$. The polynomial

$$Q(Y) = \left(\prod_{j=1}^{m} a_j^{N_j D/d_j}\right) \prod_{\{\sigma\}} P(X_0, \alpha_1^\sigma, \ldots, \alpha_m^\sigma)$$

is not identically zero, and has coefficients in $\mathbf{Z}$.

Further,

$$M(Q) = \left(\prod_{j=1}^{m} a_j^{N_jD/d_j}\right) \prod_{\{\sigma\}} \exp\left(\int_0^1 \operatorname{Log}|P(e^{2i\pi u}, \alpha_1^\sigma, \dots, \alpha_m^\sigma)|\,du\right)$$

$$\leqslant \left(\prod_{j=1}^{m} a_j^{N_jD/d_j}\right) \prod_{\{\sigma\}} \left(L(P)\prod_{j=1}^{m} \max(1, |\alpha_j^\sigma|)^{N_j}\right)$$

$$\leqslant L(P)^D . \prod_{j=1}^{m} M(\alpha_j)^{N_jD/d_j}.$$

Furthermore,

$$|Q(t)| \leqslant |P(t, \alpha_1, \dots, \alpha_m)| \left(\prod_{j=1}^{m} a_j^{N_jD/d_j}\right) \prod_{\substack{\{\sigma\}\\ \sigma\neq 1}} |P(t, \alpha_1^\sigma, \dots, \alpha_m^\sigma)|$$

$$\leqslant |P(t, \alpha_1, \dots, \alpha_m)| . L(P)^{D-1} . \max\{1, |t|\}^{N_0(D-1)}$$

$$. \exp\left\{\sum_{j=1}^{m} N_j h(\alpha_j)\right\}.$$

Let γ be a root of Q which is at minimal distance from t, and let k be its multiplicity. Then (see, for example, the proof of Lemma 2.3 of [Wa], or [M–W] Lemma 9)

$$M(\gamma)^k \leqslant M(Q),$$

$$[\mathbf{Q}(\gamma) : \mathbf{Q}] \leqslant DN_0/k$$

and

$$|\gamma - t|^k \leqslant 4^{(DN_0)^2}(2DN_0 H(Q))^{DN_0} |Q(t)|.$$

Since, for n integer $\geqslant 1$,

$$4^{n^2}(2n)^n . 2^{n^2} \leqslant 2^{4n^2},$$

and since

$$H(Q) \leqslant 2^{DN_0} M(Q),$$

the desired result follows.

We will use only a weaker form of Lemma 2.3.

COROLLARY 2.4. *With the notations of Lemma* 2.3, *we have*

$$[\mathbf{Q}(\gamma) : \mathbf{Q}] \leqslant DN_0,$$

$$H(\gamma) \leqslant (H_0 H_1 \dots H_m)^{C_3}$$

and

$$|\gamma - t| \leqslant |P((t, \alpha_1, \dots, \alpha_m)| . (H_0 H_1 \dots H_m)^{C_4},$$

where

$$H_0 = \max\{e, H(P)\}, \quad H_j = \max\{e, H(\alpha_j)\} \quad (1 \leqslant j \leqslant m),$$

and C_3, C_4 *depend only on* $D, N_0, \dots, N_m$.

LEMMA 2.5. *Let $P \in \mathbf{C}[X_1, \ldots, X_m]$ be a polynomial of total degree at most N, and let $x_1, \ldots, x_m, y_1, \ldots, y_m$ be complex numbers. Then*

$$|P(x_1, \ldots, x_m) - P(y_1, \ldots, y_m)| \leqslant NL(P) R^{N-1} \sum_{j=1}^{m} |x_j - y_j|.$$

with

$$R = \max\{1, |x_1|, \ldots, |x_m|, |y_1|, \ldots, |y_m|\}.$$

PROOF. Straightforward, using the identity

$$x_1^{h_1} \ldots x_m^{h_m} - y_1^{h_1} \ldots y_m^{h_m} = \sum_{j=1}^{m} (x_j - y_j) x_1^{h_1} \ldots x_{j-1}^{h_{j-1}} y_{j+1}^{h_{j+1}} \ldots y_m^{h_m} \sum_{k=0}^{h_j - 1} x_j^k y_j^{h_j - k - 1}.$$

LEMMA 2.6. *Let $\psi: \mathbf{R}_+ \to \mathbf{R}_+$ be a positive function defined over the set $\mathbf{R}_+$ of positive real numbers, such that*

$$\lim_{x \to +\infty} \psi(x)/x = +\infty.$$

Let $\theta_0, \ldots, \theta_m$ be complex numbers, and N a positive integer. There exist two easily computable numbers C_5 and H_0, depending only on m, ψ, N and $\max_{0 \leqslant j \leqslant m} |\theta_j|$, with the following property.

Let H be an integer with $H \geqslant H_0$, let $\xi_1, \ldots, \xi_m$ be algebraic numbers of degree at most N and height at most H, and let $P \in \mathbf{Z}[X_0, \ldots, X_m]$ be a polynomial of degree at most N and height at most H, such that the polynomial $P(X_0, \xi_1, \ldots, \xi_m) \in \mathbf{C}[X_0]$ is not identically zero. Assume

$$|\theta_1 - \xi_1| + \ldots + |\theta_m - \xi_m| + |P(\theta_0, \theta_1, \ldots, \theta_m)| < \exp\{-\psi(\operatorname{Log} H)\}.$$

Then there exists an algebraic number ξ_0 of degree at most N^{m+1} and height at most H^{C_5}, such that

$$|\theta_0 - \xi_0| \leqslant \exp\{-\tfrac{1}{2}\psi(\operatorname{Log} H)\}.$$

PROOF OF LEMMA 2.6. By Lemma 2.5, we have

$$\begin{aligned} &|P(\theta_0, \xi_1, \ldots, \xi_m) - P(\theta_0, \theta_1, \ldots, \theta_m)| \\ &\qquad \leqslant (N+1)^{m+2} (1 + \max_{0 \leqslant h \leqslant m} |\theta_h|)^N . H . \exp\{-\psi(\operatorname{Log} H)\}. \end{aligned}$$

Since $\psi(x)/x$ tends to infinity, for H sufficiently large we obtain

$$|P(\theta_0, \xi_1, \ldots, \xi_m)| \leqslant \exp\{-\tfrac{2}{3}\psi(\operatorname{Log} H)\}.$$

Using Corollary 2.4, we find an algebraic number ξ_0 of degree at most N^{m+1} and height at most H^{C_3} such that

$$\begin{aligned} |\theta_0 - \xi_0| &\leqslant |P(\theta_0, \xi_1, \ldots, \xi_m)| . H^{mC_4} \\ &\leqslant \exp\{-\tfrac{1}{2}\psi(\operatorname{Log} H)\} \quad \text{for } H \geqslant H_0. \end{aligned}$$

Another proof of Lemma 2.6 is given in [Bi] Lemma 4.5.

LEMMA 2.7. *Let $\psi\colon \mathbf{R}_+ \to \mathbf{R}_+$ be a positive function such that*

$$\lim_{x \to +\infty} \psi(x)/x = +\infty$$

and

$$\psi(2x)/\psi(x) \text{ is bounded when } x \to +\infty.$$

Let $\theta_0, \ldots, \theta_k, \omega_1, \ldots, \omega_h$ be complex numbers, and N a positive integer. There exist two numbers H_0, C_6, depending only on k, h, ψ, N, $\theta_0, \ldots, \theta_k$, $\omega_1, \ldots, \omega_h$ with the following property.

Assume that for all algebraic numbers $\alpha_0, \ldots, \alpha_k$ of degree at most N^{k+h+1} and height at most H, with $H \geqslant H_0$, we have

$$|\theta_0 - \alpha_0| + \ldots + |\theta_k - \alpha_k| > \exp\{-\psi(\operatorname{Log} H)\}.$$

Assume that there exist an integer $H_1 \geqslant H_0$, and algebraic numbers $\xi_1, \ldots, \xi_m$, with $m = h+k$, of degree at most N and height at most H_1, such that

$$|\theta_1 - \xi_1| + \ldots + |\theta_k - \xi_k| + |\omega_1 - \xi_{k+1}| + \ldots + |\omega_h - \xi_m| < \exp\{-C_6 \psi(\operatorname{Log} H_1)\}.$$

Then θ_0 is transcendental over the field $Q(\theta_1, \ldots, \theta_k, \omega_1, \ldots, \omega_h)$.

PROOF OF LEMMA 2.7. Let $x_1, \ldots, x_q$ be an algebraically independent subset of $\{\theta_1, \ldots, \theta_k, \omega_1, \ldots, \omega_h\}$, and, for $1 \leqslant j \leqslant q$, let $\eta_j \in \{\xi_1, \ldots, \xi_m\}$ be such that

$$\sum_{j=1}^{q} |x_j - \eta_j| < \exp\{-C_6 \psi(\operatorname{Log} H_1)\}.$$

Let $P \in \mathbf{Z}[X_0, X_1, \ldots, X_q]$ be a polynomial such that $P(\theta_0, x_1, \ldots, x_q) = 0$. By Lemma 2.5 we have

$$|P(\theta_0, \eta_1, \ldots, \eta_q)| \leqslant \exp\{-\tfrac{1}{2} C_6 \psi(\operatorname{Log} H_1)\}.$$

From Lemma 2.6 and from our assumption that $\theta_0, \theta_1, \ldots, \theta_k$ cannot be approximated simultaneously by algebraic numbers of bounded degree, we conclude

$$P(X_0, \eta_1, \ldots, \eta_q) \equiv 0.$$

Let us write

$$P(X_0, X_1, \ldots, X_q) = \sum_{l=0}^{N} p_l(X_1, \ldots, X_q)\, X_0^l,$$

where $p_l(X_1, \ldots, X_q) \in \mathbf{Z}[X_1, \ldots, X_q]$. Since

$$p_l(\eta_1, \ldots, \eta_q) = 0 \quad \text{for } 0 \leqslant l \leqslant N,$$

we deduce from Lemma 2.5

$$|p_l(x_1, \ldots, x_q)| \leqslant \exp\{-\psi(\operatorname{Log} H_1)\} \quad (0 \leqslant l \leqslant N).$$

The left-hand side does not depend on H_1, and therefore for $H_1 \geqslant H_0$, where H_0 depends on P,

$$p_l(x_1, \ldots, x_q) = 0 \quad \text{for } 0 \leqslant l \leqslant N.$$

This proves that

$$P(X_0, \ldots, X_q) \equiv 0.$$

3. On the difference between an algebraic number and the exponential of an algebraic number

In 1953, Mahler [Ma 2] proved that if m and p are positive integers, then

$$|e^m - p| > \exp\{-40(\operatorname{Log} m)(\operatorname{Log} p)\}.$$

In 1967 [Ma 3], he succeeded to replace 40 by 33, and in 1973, Mignotte [Mi] replaced it by 17.7. It is not yet known whether there exists an absolute constant C_7 such that

$$|e^m - p| > p^{-C_7}$$

for all positive integers m and p.

From Theorem 1.1 we deduce a lower bound for $e^\alpha - \beta$, when α and β are any non-zero algebraic numbers:

$$|e^\beta - \alpha| > \exp\{-2^{42} D^3 (\operatorname{Log} A)(\operatorname{Log} B + \operatorname{Log}\operatorname{Log} A)\},$$

where $D = [Q(\alpha, \beta) : Q]$, $A = \max\{H(\alpha), e^e\}$ and $B = H(\beta)$.

For instance if m, n, p, q are positive integers with $p \geqslant 3$, then

$$|e^{m/n} - p/q| > \exp\{-2^{42}(\operatorname{Log} p)(\operatorname{Log} m + \operatorname{Log} n + \operatorname{Log}\operatorname{Log} p)\}. \tag{3.1}$$

This result can be improved when m is relatively small: if $m < n(\operatorname{Log} p)^{1-\varepsilon}$ with $\varepsilon > 0$, then

$$|e^{m/n} - p/q| > \exp\left\{-C_8(\varepsilon)(\operatorname{Log} p)\left(1 + \frac{\operatorname{Log} n}{\operatorname{Log}\operatorname{Log} p}\right)\right\}, \tag{3.2}$$

where $C_8(\varepsilon)$ is an easily computable constant depending only on ε. More precisely, the case $n = 1$ of Theorem 1.1 shows that

$$|e^\beta - \alpha| > \exp\left\{-2^{40} . D^3(\operatorname{Log} A_1)\left(1 + \frac{\operatorname{Log} B + \operatorname{Log}\operatorname{Log} A_1}{1 + \operatorname{Log}\operatorname{Log} A_1 - \operatorname{Log}|\beta|}\right)\right\},$$

where $A_1 = \max\{H(\alpha), \exp|\beta|, e\}$.

We can replace e by e^π in (3.1): let m, n, p, q be positive integers, with $p \geqslant 3$. Then

$$|e^{\pi m/n} - p/q| > \exp\{-2^{71} . (\operatorname{Log} p)(\operatorname{Log} m + \operatorname{Log} n + \operatorname{Log}\operatorname{Log} p)\}.$$

This result can be deduced from Theorem 1.1 applied to the linear form

$$-i\frac{m}{n}\log\alpha_1-\log\frac{p}{q}$$

with $\log\alpha_1 = i\pi$, $A_1 = e^\pi$, $A_2 = p$, $D = 2$, $B = mn$ and $E_1 = e$.

The corresponding statement (3.2) for e^π is not yet known, even for $m = n = 1$.

4. On the simultaneous approximation of a complex number and its exponential

Let a be a non-zero complex number, and ξ_0, ξ_1 two algebraic numbers of height at most H_0, H_1 respectively, $H_j \geqslant e^e$. In [C], Cijsouw proved

$$|a-\xi_0|+|e^a-\xi_1| > \exp\{-C_9(\operatorname{Log} H_1)(\operatorname{Log} H_0)\}$$

and

$$|a-\xi_0|+|e^a-\xi_1| > \exp\{-C_{10}(\operatorname{Log} H)^2(\operatorname{Log}\operatorname{Log} H)^{-1}\},$$

where $H = \max\{H_1, H_0\}$, and C_9, C_{10} depend only on a and $[Q(\xi_0, \xi_1): Q]$.

Here we prove a slightly more general result.

THEOREM 4.1. *Let a be a non-zero complex number, and ξ_0, ξ_1 be two algebraic numbers. Let D, H_0, H_1 satisfy*

$$D \geqslant [Q(\xi_0, \xi_1): Q], \quad H_0 \geqslant H(\xi_0), \quad H_1 \geqslant \max\{H(\xi_1), e^e\}.$$

Then

$$|a-\xi_0|+|e^a-\xi_1| \geqslant \exp\left\{-C_{11}(a)\, D^3(\operatorname{Log} H_1)\left(1+\frac{\operatorname{Log} H_0}{\operatorname{Log}\operatorname{Log} H_1}\right)\right\},$$

where

$$C_{11}(a) = 2^{43}(1+|a|)^2.$$

PROOF OF THEOREM 4.1. There is no loss of generality to assume $|e^a-\xi_1| \leqslant \frac{1}{3}|e^a|$. By Lemma 2.2 we can choose $\log\xi_1$ such that

$$|a-\log\xi_1| \leqslant \tfrac{3}{2}|e^{-a}|.|e^a-\xi_1|.$$

Thus

$$|\xi_0-\log\xi_1| \leqslant (1+\tfrac{3}{2}|e^{-a}|)(|a-\xi_0|+|e^a-\xi_1|)$$

and

$$|\log\xi_1| \leqslant \tfrac{1}{2}+|a|.$$

From 1.2 we conclude

$$C_{11}(a) \leqslant \tfrac{6}{5}C_2(1, R) \quad \text{with} \quad R = \tfrac{3}{2}+|a|.$$

Finally, we remark that $R(2+\operatorname{Log} R)^2 \leqslant 10(1+|A|)^2$.

5. On Franklin Schneider's theorem

Let a and b be two complex numbers, with $a \neq 0$, and let $\log a$ be a non-zero determination of the logarithm of a. We consider lower bounds for

$$|a-\alpha|+|b-\beta|+|a^b-\gamma|,$$

when α, β, γ are algebraic numbers. From the work of Bijlsma [Bi] we know that this number can be very small when β is rational, and we will consider here only the case of irrational β. This problem has been studied by Ricci, Franklin, Schneider, Smelev, Bundschuh, and more recently in [Bi], [C–W], [M–W] and [Wü]. The best known results were firstly [C–W]:

$$\exp\{-C_{12}(\operatorname{Log} H)^3 \operatorname{Log}\operatorname{Log} H\},$$

where H is an upper bound for the heights of α, β, γ and C_{12} depends on $\log a, b$, and $D = [Q(\alpha,\beta,\gamma): Q]$, and secondly [M–W]:

$$\exp\{-C_{13} D^4(\operatorname{Log} H)^4 (\operatorname{Log}\operatorname{Log} H)^{-1}\},$$

where C_{13} depends only on $\log a$ and b. (This last result has been slightly improved with respect to D; see [M–W]). Here we get a sharpening of these two estimates:

THEOREM 5.1. *Let a and b be two complex numbers with $a \neq 0$, and let $\log a$ be any non-zero determination of the logarithm of a.*

Let α, β, γ be algebraic numbers of height at most H, with $H \geqslant e^e$, and let D be the degree of the field $Q(\alpha,\beta,\gamma)$ over Q. Assume that β is irrational. Then

$$|a-\alpha|+|b-\beta|+|a^b-\gamma| > \exp\{-C_{14} D^4(\operatorname{Log} H)^3 (\operatorname{Log}\operatorname{Log} H)^{-2}\}$$

with

$$C_{14} = C_{14}(\log a, b) \leqslant 2C_2(2, \tfrac{3}{2}+|\log a|+|b \log a|).$$

Several generalizations of the Franklin Schneider problems have been studied by Wallisser, Meyer, Bundschuh, and more recently in [C–W], [Bi] and [Wü]. Our Theorem 1.1 leads to several improvements of these results. Here is one example, which generalizes Theorem 5.1.

THEOREM 5.2. *Let $a_1, \ldots, a_m, b_0, \ldots, b_m$ be complex numbers, with $a_j \neq 0$ $(1 \leqslant j \leqslant m)$, and let $\log a_j$ denote an arbitrary value of the logarithm of a_j such that $\log a_j \neq 0$. Define*

$$R = \tfrac{3}{2} + \sum_{j=1}^{m} |\log a_j| + |b_0| + \sum_{j=1}^{m} |b_j \log a_j| \quad \text{and} \quad C_{15} = 2C_2(m+1, R).$$

Let $\alpha_1, \ldots, \alpha_m, \beta_0, \ldots, \beta_m, \gamma$ be algebraic numbers of height $\leqslant H$, with $H \geqslant e^e$, generating a field of degree D. Assume either $b_0 \neq 0$ or $1, \beta_1, \ldots, \beta_m$ linearly independent *over Q. Then*

$$\sum_{j=1}^{m}|a_j-\alpha_j|+\sum_{j=0}^{m}|b_j-\beta_j|+|e^{b_0}a_1^{b_1}\dots a_m^{b_m}-\gamma|$$
$$>\exp\{-C_{15}D^{m+3}(\operatorname{Log}H)^{m+2}(\operatorname{Log}\operatorname{Log}H)^{-m-1}\}.$$

PROOF OF THEOREM 5.2. This result is a straightforward consequence of Lemma 2.2 and Corollary 1.2, with $n=m+1$, $H_1=\dots=H_n=B=H$.

6. Simultaneous approximations and algebraic independence

The first connection between diophantine approximations and algebraic independence goes back to Mahler in 1932 [Ma 1]. More recent results have been obtained in special cases by Bijlsma [Bi], Laurent [L] and Väänänen [V].

In [V], Väänänen gives a lower bound for $|a-\xi_0|+|b-\xi_1|+|P(a,e^b)|$ in terms of the heights of ξ_0, ξ_1 and P, when a and b are non-zero complex numbers. Similarly, in his thesis [Bi], Bijlsma gives lower bounds for several expressions like

$$|a-\xi_0|+|b-\xi_1|+|P(a,b,a^b)-\xi_2|$$

in terms of the heights of ξ_0, ξ_1, ξ_2 and P, when a, b are non-zero complex numbers.

As remarked in [V], Väänänen's result shows that if a and b are Liouville numbers of a certain type, then a and e^b are algebraically independent. In [L], Laurent shows that the lower bound for linear forms of [M–W] implies the following result of Feldman: if a is a Liouville number of a certain type, and if β is algebraic irrational, then a and a^β are algebraically independent.

We give here a rather straightforward consequence of Theorems 4.1 and 5.2 and Lemma 2.6.

THEOREM 6.1. *Let* $m\geqslant 0$, $h\geqslant 0$ *be non-negative integers,* $a_1,\dots,a_m$, $b_0,\dots,b_m$, $c_1,\dots,c_h$ *be complex numbers, and N a positive integer. There exists an easily computable number* C_{16}, *depending only on m, h, N and*

$$\max\{|\log a_1|,\dots,|\log a_m|,|b_0|,\dots,|b_m|,|c_1|,\dots,|c_h|\},$$

with the following property.

Let H be an integer, $H\geqslant e^e$, *let* $\alpha_1,\dots,\alpha_m$, $\beta_0,\dots,\beta_m$, $\gamma_1,\dots,\gamma_h$ *be algebraic numbers of degree at most N and height at most H, and let* $P\in\mathbf{Z}[X_0,\dots,X_{2m+h+1}]$ *be a polynomial of degree at most N and height at most H. We assume either* $b_0\neq 0$ *or* $1,\beta_1,\dots,\beta_m$ **Q**-*linearly independent. Moreover, we assume that the polynomial*

$$P(X_0,\alpha_1,\dots,\alpha_m,\beta_0,\dots,\beta_m,\gamma_1,\dots,\gamma_h)\in\mathbf{C}[X_0]$$

is not identically zero.

Then

$$\sum_{i=1}^{m}|a_i-\alpha_i|+\sum_{j=0}^{m}|b_j-\beta_j|+\sum_{k=1}^{h}|c_k-\gamma_k|$$
$$+|P(e^{b_0}a_1^{b_1}\dots a_m^{b_m}, a_1,\dots,a_m, b_0,\dots,b_m, c_1,\dots,c_h)|$$
$$>\exp\{-C_{16}(\operatorname{Log} H)^{m+2}(\operatorname{Log}\operatorname{Log} H)^{-m-1}\}.$$

A more careful estimation of the constants involved in the proofs of Lemma 2.6 and Theorem 4.1 shows the following. Let $P\in \mathbf{Z}[X, Y]$ be a polynomial of height at most H_0 and degree at most d_0, d_0' with respect to X, Y. Let x, y be complex numbers, and α, β be algebraic numbers of degree at most d_1, d_2 and height at most H_1, H_2 respectively, satisfying $P(X,\beta)\neq 0$ and $\alpha\neq 0$. For convenience we assume $H_0\geqslant 16$ and $H_1\geqslant 16$. Then

$$|x-\alpha|+|y-\beta|+|P(e^x,y)|$$
$$>\exp\left\{-C_{17}(\operatorname{Log} H_0+\operatorname{Log} H_2)\left(1+\frac{\operatorname{Log} H_1}{\operatorname{Log}\operatorname{Log} H_0+\operatorname{Log}\operatorname{Log} H_2}\right)\right\},$$

where

$$C_{17}=2^{45}.(1+|x|)^2(d_0+d_0')(d_0d_1)^3d_2^4.$$

(Compare with [V].)

Similarly, it is easy to derive from the previous estimates the following results. Let a_1, a_2, b be complex numbers, and N a positive integer. Assume $a_2\neq 0$, and let $\log a_2$ be a determination of the logarithm of a_2. Let $\alpha_1, \alpha_2, \beta$ be algebraic numbers of degree at most N and height at most H_1, H_2, H_3 respectively. Let $P\in\mathbf{Z}[X, Y]$ be a polynomial of (total) degree at most N and height at most H_0. Assume

$$H_i\geqslant 16\quad (i=0,1,2),\qquad \beta\notin Q\qquad P(\alpha_1, Y)\neq 0.$$

We define

$$H^*=\max\{H_0H_1, H_2\},\quad H_*=\min\{H_0H_1, H_2\}$$

and

$$C_{18}=2^{67}.(1+|\log a_2|)^2.(1+|b\log a_2|)^2N^{19}.$$

Then

$$|a_1-\alpha_1|+|a_2-\alpha_2|+|b-\beta|+|P(a_1,a_2^b)|$$
$$>\exp\{-C_{18}(\operatorname{Log} H_0+\operatorname{Log} H_1)(\operatorname{Log} H_2)(\operatorname{Log} H_3+\operatorname{Log}\operatorname{Log} H^*)(\operatorname{Log}\operatorname{Log} H_*)^{-2}\}.$$

Finally, we give a result of algebraic independence which is an easy consequence of Theorem 6.1 (see Lemma 2.7).

THEOREM 6.2. *Let $m\geqslant 0$, $h\geqslant 0$ be non-negative integers, $a_1,\dots,a_m$, $b_0,\dots,b_m$, $c_1,\dots,c_h$ be complex numbers, and N a positive integer.*

Assume that there exists an increasing sequence H_l of positive integers and, for each l, that there exist algebraic numbers $\alpha_1^{(l)}, \dots, \alpha_m^{(l)}$, $\beta_0^{(l)}, \dots, \beta_m^{(l)}$, $\gamma_1^{(l)}, \dots, \gamma_h^{(l)}$, of degree at most N and height at most H_l, such that

$$\sum_{i=1}^{m} |a_i - \alpha_i^{(l)}| + \sum_{j=0}^{m} |b_j - \beta_j^{(l)}| + \sum_{k=1}^{h} |c_k - \gamma_k^{(l)}| < \exp\{-(\operatorname{Log} H_l)^{m+2}\}.$$

Assume moreover either $b_0 \neq 0$, or $1, \beta_1^{(l)}, \dots, \beta_m^{(l)}$ $\mathbf{Q}$-linearly independent for each l. Then the number $e^{b_0} a_1^{b_1} \dots a_m^{b_m}$ is transcendental over the field

$$\mathbf{Q}(a_1, \dots, a_m, b_0, \dots, b_m, c_1, \dots, c_h).$$

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