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An introduction to RAMANUJAN's mathematics

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http://www.math.jussieu.fr/ miw/

Biography of Srinivasa Ramanujan

(December 22, 1887 — April 26, 1920)

1887: born in Erode (near Tanjore)

1894-1903: school in Kumbakonam

In 1900 he began to work on his own on mathematics summing geometric and arithmetic series.

1903: G.S.Carr - *A synopsis of elementary results — a book on pure mathematics* (1886)
$5\,000$ formulae

Town High School, Kumbakonam

$$\sqrt{x}+y=7, \qquad x+\sqrt{y}=11$$

$$x=9, \qquad y=4.$$

Biography *(continued)*

1903 (December): exam at Madras University

1904 (January): enters Government Arts College, Kumbakonam

Sri K. Ranganatha Rao Prize
Subrahmanyam scholarship

MacTutor History of Mathematics

http://www-history.mcs.st-andrews.ac.uk/Mathematicians/Ramanujan.html

By 1904 Ramanujan had begun to undertake deep research. He investigated the series $\sum 1/n$ and calculated Euler's constant to 15 decimal places. He began to study the Bernoulli numbers, although this was entirely his own independent discovery.

$$S_N=\sum_{n=1}^{N}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{N}$$

$$\int_1^N\frac{dx}{x+1}<S_N<1+\int_1^N\frac{dx}{x}$$

$$C=\lim_{N\to\infty}(S_N-\log N).$$

(Euler constant)

Bernoulli numbers

$$B_0 = 1, \qquad \sum_{k=0}^{n-1} \binom{n}{k} B_k = 0 \quad \text{for } n > 1.$$

$$B_0 + 2B_1 = 0 \qquad B_1 = -\frac{1}{2}$$

$$B_0 + 3B_1 + 3B_2 = 0 \qquad B_2 = \frac{1}{6}$$

$$B_0 + 4B_1 + 6B_2 + 4B_3 = 0 \qquad B_3 = 0$$

$$B_0 + 5B_1 + 10B_2 + 10B_3 + 5B_4 = 0 \qquad B_4 = -\frac{1}{30}$$

$$\vdots$$

1905: Fails final exam

1906: Enters Pachaiyappa's College, Madras
III, goes back to Kumbakonam

1907 (December): Fails final exam.

1908: continued fractions and divergent series

1909 (April): underwent an operation

1909 (July 14): marriage with S Janaki Ammal
(1900—1994)

1910: meets Ramaswami Aiyar

1911: first mathematical paper

1912: clerk office, Madras Port Trust

Sir Francis Spring and Sir Gilbert Walker get a scholarship for him from the University of Madras starting May 1913 for 2 years.

1912: Questions in the
Journal of the Indian Mathematical Society

$$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+\cdots}}}} = ?$$

$$\sqrt{6+2\sqrt{7+3\sqrt{8+4\sqrt{9+\cdots}}}} = ?$$

Answers from Ramanujan

$$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+\cdots}}}} = 3$$

$$\sqrt{6+2\sqrt{7+3\sqrt{8+4\sqrt{9+\cdots}}}} = 4$$

$$(n+2)^2 = 1 + (n+1)(n+3)$$

$$n(n+2) = n\sqrt{1+(n+1)(n+3)}$$

$$f(n) = n(n+2)$$

$$f(n) = n\sqrt{1+f(n+1)}$$

$$f(n) = n\sqrt{1+(n+1)\sqrt{1+f(n+2)}}$$

$$= n\sqrt{1+(n+1)\sqrt{1+(n+2)\sqrt{1+(n+3)\cdots}}}$$

$$f(1) = 3$$

$$(n+3)^2 = n+5+(n+1)(n+4)$$
$$n(n+3) = n\sqrt{n+5+(n+1)(n+4)}$$
$$g(n) = n(n+3)$$
$$g(n) = n\sqrt{n+5+g(n+1)}$$
$$g(n) = n\sqrt{n+5+(n+1)\sqrt{n+6+g(n+2)}}$$
$$= n\sqrt{n+5+(n+1)\sqrt{n+6+(n+2)\sqrt{n+7}}}$$

$$g(1) = 4$$

Letter from Ramanujan to M.J.M. Hill in 1912

$$1+2+3+\cdots+\infty = -\frac{1}{12}$$
$$1^2+2^2+3^2+\cdots+\infty^2 = 0$$
$$1^3+2^3+3^3+\cdots+\infty^3 = \frac{1}{240}$$

Answer from M.J.M. Hill in 1912

$$1+2+3+\cdots+n = \frac{1}{2}n(n+1)$$
$$1^2+2^2+3^2+\cdots+n^2 = \frac{1}{3}n(n+1/2)(n+1)$$
$$1^3+2^3+3^3+\cdots+n^3 = \left(n(n+1)/2\right)^2$$

Renormalisation of divergent series (Euler,. . .)

Letters to H.F.Baker and E.W.Hobson in 1912: no answers. . .

First letter of Ramanujan to Hardy
(January 16, 1913)

I have had no university education but I have undergone the ordinary school course. After leaving school I have been employing the spare time at my disposal to work at mathematics. I have not trodden through the conventional regular course which is followed in a university course, but I am striking out a new path for myself. I have made a special investigation of divergent series in general and the results I get are termed by the local mathematicians as "startling".

$$1-2+3-4+\cdots = -\frac{1}{4}$$
$$1-1!+2!-3!+\cdots = .596\cdots$$

Answer from Hardy (February 8, 1913)

I was exceedingly interested by your letter and by the theorems which you state. You will however understand that, before I can judge properly of the value of what you have done, it is essential that I should see proofs of some of your assertions. Your results seem to me to fall into roughly three classes:

(1) there are a number of results that are already known, or easily deducible from known theorems;

(2) there are results which, so far as I know, are new and interesting, but interesting rather from their curiosity and apparent difficulty than their importance;

(3) there are results which appear to be new and important. . .

1913, February 27:
New letter from Ramanujan to Hardy

1913: Visit of Neville to India

1914, March 17 to April 14: travel to Cambridge.

1918: (May) Fellow of the Royal Society
(November) Fellow of Trinity College, Cambridge.

1919, February 27 to March 13: travel back to India.

RAMANUJAN – TAXI CAB NUMBER

$$1729 = 1^3 + 12^3 = 9^3 + 10^3$$

Euler:

$$59^4 + 158^4 = 133^4 + 134^4 = 635\,318\,657$$

RAMANUJAN – TAXI CAB NUMBER

$$1729 = 1^3 + 12^3 = 9^3 + 10^3$$

$$\begin{aligned} 4\,104 &= 2^3 + 16^3 = 9^3 + 15^3 \\ 13\,832 &= 2^3 + 24^3 = 18^3 + 20^3 \\ 40\,033 &= 9^3 + 34^3 = 16^3 + 33^3 \end{aligned}$$

Diophantine equations

$$x^3 + y^3 + z^3 = w^3$$

$(x, y, z, w) = (3, 4, 5, 6)$:
$3^3 + 4^3 + 5^3 = 27 + 64 + 125 = 216 = 6^3$

Parametric solution:

$$\begin{aligned} x &= 3a^2 + 5ab - 5b^2 \\ y &= 4a^2 - 4ab + 6b^2 \\ z &= 5a^2 - 5ab - 3b^2 \\ w &= 6a^2 - 4ab + 4b^2 \end{aligned}$$

RAMANUJAN – NAGELL EQUATION

$$x^2 + 7 = 2^n$$

$$\begin{aligned} 1^2 + 7 &= 2^3 = 8 \\ 3^2 + 7 &= 2^4 = 16 \\ 5^2 + 7 &= 2^5 = 32 \\ 11^2 + 7 &= 2^7 = 128 \\ 181^2 + 7 &= 2^{15} = 32\,768 \end{aligned}$$

Nagell (1948): no further solution

Apéry (1960): for $D > 0$, $D \neq 7$, the equation $x^2 + D = 2^n$ has at most 2 solutions.

Examples with 2 solutions:

$D = 23:$ $3^2 + 23 = 32,$ $45^2 + 23 = 2^{11} = 2$

$D = 2^{\ell+1} - 1, \ell \geq 3$: $(2^\ell - 1)^2 + 2^{\ell+1} - 1 = 2^{2\ell}$

Beukers (1980): at most one solution otherwise.

M. Bennett (1995): considers the case $D < 0$.

Partitions

$$\begin{aligned} &1 && p(1) = 1 \\ &2 = 1 + 1 && p(2) = 2 \\ &3 = 2 + 1 = 1 + 1 + 1 && p(3) = 3 \\ &4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 && \\ & = 1 + 1 + 1 + 1 && p(4) = 5 \end{aligned}$$

$$p(5) = 7, \quad p(6) = 11, \quad p(7) = 15, \ldots$$

MacMahon: table of the first 200 values

Ramanujan:

$p(5n+4)$	5
$p(7n+5)$	7
$p(11n+6)$	11
$p(25n+24)$	25
$p(49n+47)$	49
$p(121n+116)$	121

Euler:

$$1+p(1)x+p(2)x^2+\cdots+p(n)x^n+\cdots$$
$$=\frac{1}{(1-x)(1-x^2)(1-x^3)\cdots(1-x^n)\cdots}$$

$$1+\sum_{n=1}^{\infty}p(n)x^n=\prod_{n=1}^{\infty}(1-x^n)^{-1}$$

Riemann zeta function:

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_p(1-p^{-s})^{-1}$$
$$x(1-x)^{-1}=\sum_{n=1}^{\infty}x^n$$

Ramanujan tau function:

$$x\prod_{n=1}^{\infty}(1-x^n)^{24}=\sum_{n=1}^{\infty}\tau(n)x^n.$$
$$\sum_{n=1}^{\infty}\frac{\tau(n)}{n^s}=\prod_p(1-\tau(p)p^{-s}+p^{11-2s})^{-1}$$

Ramanujan's Congruences:

$\tau(pn)$ $\quad p \quad p=2,\ 3,\ 5,\ 7,\ 23.$

also: congruences modulo 691
(numerator of Bernoulli number B_{12})

Ramanujan's Conjecture, proved by Deligne in 1974

$$|\tau(p)|<2p^{11/2}$$

Hardy-Ramanujan:
for almost all integers n, the number of prime factors of n is $\log\log n$.

$A_\epsilon(x)=\#\{n\le x\ ;$
$(1-\epsilon)\log\log n<\omega(n)<(1+\epsilon)\log\log$

$$\frac{1}{x}A_\epsilon(x)\rightarrow 1 \qquad x\rightarrow\infty.$$

Highly composite numbers
(Proc. London Math. Soc. 1915)

$n=$	2	4	6	12	24	36	48	60	120..
$d(n)=$	2	3	4	6	8	9	10	12	16..

Approximation for π due to Ramanujan:

$$\frac{63}{25}\left(\frac{17+15\sqrt{5}}{7+15\sqrt{5}}\right) = 3.141592653\ 80568820189839000630$$

$$\pi = 3.141592653\ 58979323846264338328$$

Another formula due to Ramanujan for π

$$\pi = \frac{9\,801}{\sqrt{8}}\left(\sum_{n=0}^{\infty}\frac{(4n)!(1\,103+26\,390n)}{(n!)^4 396^{4n}}\right)^{-1}$$

$n=0$: 6 exact digits for $3,141592\ldots$

$n \to n+1$: 8 more digits

used in 1985: $1.7\cdot 10^7$ digits for π (1.7 crores)
Remark. In 1999: $2\cdot 10^{10}$ digits ($2\,000$ crores)

Ramanujan's formula for $1/\pi$

$$\frac{1}{\pi} = \sum_{m=0}^{\infty}\binom{2m}{m}\frac{42m+5}{2^{12m+4}}.$$

Ramanujan Notebooks

Written from 1903 to 1914

First: 16 chapters, 134 pages
Second: 21 chapters, 252 pages
Third: 33 pages

B.M. Wilson, G.N.Watson

Edited in 1957 in Bombay

The lost notebook: George Andrews, 1976

Bruce Berndt, 1985–87 (5 volumes)

Last work of Ramanujan:

Mock theta functions

References

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