Diophantine analysis and words

Michel Waldschmidt

Institut de Mathématiques de Jussieu + CIMPA
http://www.math.jussieu.fr/∼miw/

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Billiards in the cube

MR1193183 (93i:11094)
Shiokawa, Iekata (J-KEIO); **Tamura, Jun-ichi**
Description of sequences defined by billiards in the cube.
Proc. Japan Acad. Ser. A Math. Sci. **68** (1992), *no. 7*, 207–211.
11K55

MR1279582 (95c:58059)
Arnoux, Pierre (F-PARIS7-M); **Mauduit, Christian** (F-LYON-MI);
Shiokawa, Iekata (J-KEIOE); **Tamura, Jun-ichi**
Rauzy's conjecture on billiards in the cube.
Tokyo J. Math. **17** (1994), *no. 1*, 211–218.
58F03 (11B99)

MR1259106 (94m:11028)
Arnoux, Pierre (F-PROVS-DM); **Mauduit, Christian** (F-LYON-LD);
Shiokawa, Iekata (J-KEIOE); **Tamura, Jun-ichi**
Complexity of sequences defined by billiard in the cube. (English, French summaries)
Bull. Soc. Math. France **122** (1994), *no. 1*, 1–12.
11B85 (05B45 58F03)

Billiards (continued)

The billiard problem in a square is well known. Consider the orbit of a billiard ball which one can code in the following way: each time the ball meets a "vertical" side [resp. "horizontal"] mark 1 [resp. 2]. An orbit is thus represented by an infinite sequence of 1's and 2's. Let $p(n)$ be the complexity of the sequence, i.e. the number of words of length n that occur in the sequence. A classical result states that either $p(n)$ is uniformly bounded if the initial direction (α, β) is rational (i.e. $\alpha/\beta \in \mathbf{Q} \cup \{\infty\}$) and the orbit is periodic, or $p(n) = n + 1$ if $\alpha/\beta \in \mathbf{R} \smallsetminus \mathbf{Q}$ (the sequence is Sturmian).

The authors discuss the similar 3-dimensional problem, i.e. the billiard problem in the cube. An orbit is now coded by an infinite sequence on three symbols. Let $(\alpha_1, \alpha_2, \alpha_3)$ be the initial direction. Suppose $\alpha_1, \alpha_2, \alpha_3$ are $\mathbf{Q}$-independent. The authors prove the beautiful result $p(n) = n^2 + n + 1$. The proof is by no means easy: it combines skillful geometry on the torus with combinatorics.

The authors conclude the paper with questions pertaining to higher dimensions. Let $P(n, s)$ be the complexity of the orbit-sequence in the s-dimensional cube. It is not known whether $P(n, s)$ depends or not on the initial direction $(\alpha_1, \alpha_2, \cdots, \alpha_s)$ even if these numbers are $\mathbf{Q}$-independent. The authors show that if $\min\{n, s\} \le 2$ then $P(n, s)$ exists and $P(n, s) = P(s, n)$. We are left with an intriguing problem: Do these results hold for all n, s?

Reviewed by *M. Mendès France*

Complexity of the g–ary expansion of an irrational algebraic real number

Let $g \geq 2$ be an integer.

- É. Borel (1909 and 1950) : *the g–ary expansion of an algebraic irrational number should satisfy some of the laws shared by almost all numbers (for Lebesgue's measure).*
- In particular *each digit should occur*, hence *each given sequence of digits should occur infinitely often.*
- There is no explicitly known example of a triple (g, a, x), where $g \geq 3$ is an integer, a a digit in $\{0, \ldots, g-1\}$ and x an algebraic irrational number, for which one can claim that the digit a occurs infinitely often in the g–ary expansion of x.

Words

- We consider an alphabet A with g letters. The free monoid A^* on A is the set of *finite words* $a_1 \ldots a_n$ where $n \geq 0$ and $a_i \in A$ for $1 \leq i \leq n$. The law on A^* is called *concatenation.*
- The number of letters of a finite word is its *length* : the length of $a_1 \ldots a_n$ is n.
- The number of words of length n is g^n for $n \geq 0$. The single word of length 0 is the empty word e with no letter. It is the neutral element for the concatenation.

Infinite words

- We shall consider *infinite words* $w = a_1 \dots a_n \dots$
  A *factor of length* m of such a w is a word of the form $a_k a_{k+1} \dots a_{k+m-1}$ for some $k \geq 1$.
- The *complexity* of an infinite word w is the function $p(m)$ which counts, for each $m \geq 1$, the number of distinct factors of w of length m.
- Hence for an alphabet A with g elements we have $1 \leq p(m) \leq g^m$ and the function $m \mapsto p(m)$ is non–decreasing.
- According to Borel's suggestion, the complexity of the sequence of digits in basis g of an irrational algebraic number should be $p(m) = g^m$.

Automatic sequences

- Let $g \geq 2$ be an integer. An infinite sequence $(a_n)_{n\geq 0}$ is said to be g–automatic if a_n is a finite-state function of the base g representation of n : this means that there exists a finite automaton starting with the g–ary expansion of n as input and producing the term a_n as output.
- A. Cobham, 1972 : *Automatic sequences have a complexity* $p(m) = O(m)$.

Morphisms

- Let A and B be two finite sets. A map from A to B^* can be uniquely extended to a homomorphism between the free monoids A^* and B^*. We call *morphism from A to B* such a homomorphism.
- Consider a morphism ϕ from A into itself for which there exists a letter a such that $\phi(a) = au$, where u is a non–empty word such that $\phi^k(u) \neq e$ for every $k \geq 0$. In that case, the sequence of finite words $(\phi^k(a))_{k\geq 1}$ converges in $A^{\mathbf{N}}$ (endowed with the product topology of the discrete topology on each copy of A) to an infinite word $w = au\phi(u)\phi^2(u)\phi^3(u)\ldots$. This infinite word is clearly a fixed point for ϕ and we say that *w is generated by the morphism ϕ*.

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Recurrent morphisms, binary morphisms, morphic sequences

- If, moreover, every letter occurring in w occurs at least twice, then we say that w is generated by a *recurrent morphism.*
- If the alphabet A has two letters, then we say that w is generated by a *binary morphism.*
- More generally, an infinite sequence w in $A^{\mathbf{N}}$ is said to be *morphic* if there exist a sequence u generated by a morphism ϕ defined over an alphabet B and a morphism from B to A such that $w = \phi(u)$.

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Example 1 : the Fibonacci word

Take $A = \{a, b\}$.

- Start with $f_1 = b$, $f_2 = a$ and define (concatenation) : $f_n = f_{n-1}f_{n-2}$.
- Hence $f_3 = ab$ $f_4 = aba$ $f_5 = abaab$
  $f_6 = abaababa$ $f_7 = abaababaabaab$
  $f_8 = abaababaabaababaababa$
- The *Fibonacci word*

$$w = abaababaabaababaababaabaababaabaab\dots$$

is generated by a binary recurrent morphism : it is the fixed point of the morphism $a \mapsto ab$, $b \mapsto a$ under this morphism, the image of f_n is f_{n+1}.

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Example 2 : the Thue-Morse sequence 01101001100101101 . . .

- For $n \geq 0$ define $a_n = 0$ if the sum of the binary digits in the expansion of n is even, $a_n = 1$ if this sum is odd : the *Thue-Morse sequence* $(a_n)_{n\geq 0}$ starts with

$$0\,1\,1\,0\,1\,0\,0\,1\,1\,0\,0\,1\,0\,1\,1\,0\,1\dots$$

- Replace 0 by a and 1 by b.
  The *Thue-Morse word*

$$w = abbabaabbaababbab\dots$$

is generated by a binary recurrent morphism : it is the fixed point of the morphism $a \mapsto ab$, $b \mapsto ba$.

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The Thue-Morse-Mahler number

- The *Thue-Morse-Mahler number in basis* $g \geq 2$ is the number
$$\xi_g = \sum_{n \geq 0} \frac{a_n}{g^n}$$
where $(a_n)_{n \geq 0}$ is the Thue-Morse sequence. The g–ary expansion of ξ_g starts with
$$0.1101001100101101\ldots$$
- These numbers were considered by K. Mahler who proved in 1929 that they are transcendental.

Example 3 : the Rudin–Shapiro sequence

- The Rudin–Shapiro word $aaabaabaaaabbbab\ldots$. For $n \geq 0$ define $r_n \in \{a,\ b\}$ as being equal to a (respectively b) if the number of occurrences of the pattern 11 in the binary representation of n is even (respectively odd).
- Let σ be the morphism defined from the monoid B^* on the alphabet $B = \{1, 2, 3, 4\}$ into B^* by : $\sigma(1) = 12$, $\sigma(2) = 13$, $\sigma(3) = 42$ and $\sigma(4) = 43$. Let
$$\mathbf{u} = 121312421213\ldots$$
be the fixed point of σ begining with 1 and let φ be the morphism defined from B^* to $\{a, b\}^*$ by : $\varphi(1) = aa$, $\varphi(2) = ab$ and $\varphi(3) = ba$, $\varphi(4) = bb$. Then the Rudin-Shapiro word is $\varphi(\mathbf{u})$, hence it is morphic.

Example 4 : the Baum-Sweet sequence

- The Baum-Sweet sequence. For $n \geq 0$ define $a_n = 1$ if the binary expansion of n contains no block of consecutive 0's of odd length, $a_n = 0$ otherwise : the sequence $(a_n)_{n\geq 0}$ starts with

$$1\,1\,0\,1\,1\,0\,0\,1\,0\,1\,0\,0\,1\,0\,0\,1\,1\,0\,0\,1\,0\ldots$$

- The *Baum-Sweet word* on the alphabet $\{a, b\}$

$$bbabbaababaabaabbaaba\ldots$$

is morphic.

Example 5 : powers of 2

The binary automatic number

$$\sum_{n\geq 0} 2^{-2^n} = 0.1101000100000001000\cdots$$

yields the word

$$\mathbf{v} = v_1 v_2 \cdots v_n \cdots = bbabaaabaaaaaaabaaa\cdots$$

where

$$v_n = \begin{cases} b & \text{if } n \text{ is a power of } 2, \\ a & \text{otherwise.} \end{cases}$$

The complexity of $p(m)$ of $\mathbf{v}$ is bounded by $2m$:

$m =$	1	2	3	4	5	6	$\cdots$
$p(m) =$	2	4	6	7	9	11	$\cdots$

Sturmian words

Assume $g = 2$, say $A = \{a, b\}$.

- A word is periodic if and only if its complexity is bounded.
- If the complexity $p(m)$ a word w satisfies $p(m) = p(m+1)$ for one value of m, then $p(m+k) = p(m)$ for all $k \geq 0$, hence the word is periodic. It follows that *a non–periodic w has a complexity* $p(m) \geq m+1$.
- An infinite word of minimal complexity $p(m) = m+1$ is called *Sturmian* (Morse and Hedlund, 1938).
- Examples of Sturmian words are given by 2–dimensional billiards.

The Fibonacci word is Sturmian

The Fibonacci word
$w = abaababaabaababaababaabaababaabaab\ldots$ is Sturmian.

- For $n \geq 2$ the word f_n is deduced from w by taking only the first F_n letters.
- There are exactly three words of two letters which occur in the sequence, namely aa, ab and ba. In other terms bb does not occur.
- There are exactly 4 factors of length 3 (while the number of words of length 3 on the alphabet with two letters is 8), namely aab, aba, baa, bab.
- There are 5 words of 4 letters (among 16 possibilities).

The Fibonacci word is Sturmian

The Fibonacci word is Sturmian.

$$
\begin{array}{ccccccccc}
 & & & & & & & & aabaa \\
 & & & & & & & \nearrow & \\
a & \to & aa & \to & aab & \to & aaba & \to & aabab \\
 & \searrow & & & & & & & \\
 & & ab & \to & aba & \to & abaa & \to & abaab \\
 & & & & & \searrow & & & \\
 & & & & & & abab & \to & ababa \\
b & \to & ba & \to & baa & \to & baab & \to & baaba \\
 & & & \searrow & & & & & \\
 & & & & bab & \to & baba & \to & babaa
\end{array}
$$

Sturmian words

On the alphabet $\{a, b\}$, a Sturmian word w is characterized by the property that *for each* $m \geq 1$, *there is exactly one factor* v *of* w *of length* m *such that both* va *and* vb *are factors of* w *of length* $m+1$.

The Fibonacci word is Sturmian

- Claim : *For each m the number $p(m)$ of factors of w of length m in w is $m+1$.*
- First step : *the Fibonacci word is not periodic*.
- Proof. The word f_n has length F_n, it consists of F_{n-1} letters a and F_{n-2} letters b. Hence the proportion of a in the Fibonacci word w is $1/\Phi$, where Φ is the Golden number
$$\Phi = \frac{1+\sqrt{5}}{2}$$
which is an irrational number.
- *Remark*. The proportion of b in w is $1/\Phi^2$ with $(1/\Phi) + (1/\Phi^2) = 1$, as expected !

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The Fibonacci word is Sturmian

Second step : for $n \geq 3$ the word f_n can be written

$$f_n = \begin{cases} u_n ba & \text{for even } n, \\ u_n ab & \text{for odd } n, \end{cases}$$

with u_n of length $F_n - 2$. By induction one checks that *the word u_n is palindromic*.

- *Proof :* for even n we have
$$u_{n+1} = f_n u_{n-1} = u_{n-1} ab u_{n-2} ba u_{n-1}$$
while for odd n the formula is
$$u_{n+1} = f_n u_{n-1} = u_{n-1} ba u_{n-2} ab u_{n-1}.$$

The Fibonacci word is Sturmian (continued)

Step 3 :

$$f_{n-1}f_n = \begin{cases} u_{n+1}ba & \text{for even } n, \\ u_{n+1}ab & \text{for odd } n, \end{cases}$$

which means that *in the two words* $f_{n-1}f_n$ *and* f_{n+1}*, only the last two letters are not the same.*

- *Proof :* write $f_{n-1}f_n$ as $f_{n-1}f_{n-1}f_{n-2}$; for even n we have $u_{n+1} = f_{n-1}u_{n-2}bau_{n-1}$ and

  $$f_{n-1}f_n = f_{n-1}u_{n-1}abu_{n-2}ba = f_{n-1}u_{n-2}bau_{n-1}ba = u_{n+1}ba,$$

  while for odd n we have $u_{n+1} = f_{n-1}u_{n-2}abu_{n-1}$ and

  $$f_{n-1}f_n = f_{n-1}u_{n-1}bau_{n-2}ab = f_{n-1}u_{n-2}abu_{n-1}ab = u_{n+1}ab.$$

The Fibonacci word is Sturmian (end)

The next step is the proof that the Fibonacci word is Sturmian is to check that the number of factors of length $F_n - 1$ in w is at most F_n.

The proof involves the factorisation of w on $\{f_{n-1}, f_{n-2}\}$.

It follows that $p(m) = m + 1$ for infinitely many m, hence for all $m \geq 1$.

Transcendence and Sturmian words

- S. Ferenczi, C. Mauduit, 1997 : A number whose sequence of digits is Sturmian is transcendental.
  Combinatorial criterion : *the complexity of the* g*–ary expansion of every irrational algebraic number satisfies*

$$\liminf_{m\to\infty}(p(m) - m) = +\infty.$$

- *Tool :* a p–adic version of the Thue–Siegel–Roth Theorem due to Ridout (1957).

Liouville numbers

- Liouville's Theorem : *for any real algebraic number* α *there exists a constant* $c > 0$ *such that the set of* $p/q \in \mathbf{Q}$ *with* $|\alpha - p/q| < q^{-c}$ *is finite.*
- Liouville's Theorem yields the transcendence of the value of a series like $\sum_{n\geq 0} 2^{-u_n}$, provided that the sequence $(u_n)_{n\geq 0}$ is increasing and satisfies

$$\limsup_{n\to\infty} \frac{u_{n+1}}{u_n} = +\infty.$$

- For instance $u_n = n!$ satisfies this condition : hence the number $\sum_{n\geq 0} 2^{-n!}$ is transcendental.

Roth's Theorem

- Roth's Theorem : *for any real algebraic number* α, *for any* $\epsilon > 0$, *the set of* $p/q \in \mathbf{Q}$ *with* $|\alpha - p/q| < q^{-2-\epsilon}$ *is finite.*
- Roth's Theorem yields the transcendence of $\sum_{n\geq 0} 2^{-u_n}$ under the weaker hypothesis
$$\limsup_{n\to\infty} \frac{u_{n+1}}{u_n} > 2.$$
- The sequence $u_n = [2^{\theta n}]$ satisfies this condition as soon as $\theta > 1$. For example the number
$$\sum_{n\geq 0} 2^{-3^n}$$
is transcendental.

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Transcendence of $\sum_{n\geq 0} 2^{-2^n}$

- A stronger result follows from Ridout's Theorem, using the fact that the denominators 2^{u_n} are powers of 2 : the condition
$$\limsup_{n\to\infty} \frac{u_{n+1}}{u_n} > 1$$
suffices to imply the transcendence of the sum of the series $\sum_{n\geq 0} 2^{-u_n}$ (cf. Florian Luca lecture on tuesday).
- Since $u_n = 2^n$ satisfies this condition, the transcendence of $\sum_{n\geq 0} 2^{-2^n}$ follows (Kempner 1916).
- Ridout's Theorem : *for any real algebraic number* α, *for any* $\epsilon > 0$, *the set of* $p/q \in \mathbf{Q}$ *with* $q = 2^k$ *and* $|\alpha - p/q| < q^{-1-\epsilon}$ *is finite.*

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Mahler's method for the transcendence of $\sum_{n\geq 0} 2^{-2^n}$

- Mahler (1930, 1969) : the function $f(z) = \displaystyle\sum_{n\geq 0} z^{-2^n}$ satisfies $f(z^2) + z = f(z)$ for $|z| < 1$.
- J.H. Loxton and A.J. van der Poorten (1982–1988).
- P.G. Becker (1994) : *for any given non–eventually periodic automatic sequence* $\mathbf{u} = (u_1, u_2, \dots)$, *the real number*

$$\sum_{k\geq 1} u_k g^{-k}$$

is transcendental, provided that the integer g *is sufficiently large (in terms of* $\mathbf{u}$*).*

More on Mahler's method

- K. Nishioka (1991) : algebraic independence measures for the values of Mahler's functions.
- For any integer $d \geq 2$,

$$\sum_{n\geq 0} 2^{-d^n}$$

is a S–number in the classification of transcendental numbers due to… Mahler.
- Reference : K. Nishioka, *Mahler functions and transcendence,* Lecture Notes in Math. **1631**, Springer Verlag, 1996.
- Conjecture – P.G. Becker, J. Shallitt : *more generally any automatic irrational real number is a* S*–number.*

Further transcendence results on g-ary expansions of real numbers

- J-P. Allouche and L.Q. Zamboni(1998).
- R.N. Risley and L.Q. Zamboni(2000).
- B. Adamczewski and J. Cassaigne (2003).

Complexity of the g-ary expansion of an algebraic number

- **Theorem** (B. Adamczewski, Y. Bugeaud, F. Luca 2004). *The binary complexity p of a real irrational algebraic number x satisfies*

$$\liminf_{m\to\infty} \frac{p(m)}{m} = +\infty.$$

- Corollary (conjecture of A. Cobham (1968)) : *If the sequence of digits of an irrational real number x is automatic, then x is transcendental.*

Irrationality measures for automatic numbers

- Further progress by B. Adamczewski and J. Cassaigne (2006) – solution to a Conjecture of J. Shallit (1999) : *A Liouville number cannot be generated by a finite automaton.*
- The irrationality measure of the automatic number associated with $\sigma(0) = 0^n1$ and $\sigma(1) = 1^n0$ is at least n.
- For the Thue-Morse-Mahler numbers for instance the exponent of irrationality is ≤ 5.

Christol, Kamae, Mendes-France, Rauzy

The result of B. Adamczewski, Y. Bugeaud and F. Luca implies the following statement related to the work of G. Christol, T. Kamae, M. Mendès-France and G. Rauzy (1980) :

Corollary. *Let $g \geq 2$ be an integer, p be a prime number and $(u_k)_{k\geq 1}$ a sequence of integers in the range $\{0, \dots, p-1\}$. The formal power series*

$$\sum_{k\geq 1} u_k X^k$$

and the real number

$$\sum_{k\geq 1} u_k g^{-k}$$

are both algebraic (over $\mathbf{F}_p(X)$ and over $\mathbf{Q}$, respectively) if and only if they are rational.

Schmidt's subspace Theorem (single place)

For $\mathbf{x} = (x_0, \ldots, x_{m-1}) \in \mathbf{Z}^m$, define $|\mathbf{x}| = \max\{|x_0|, \ldots, |x_{m-1}|\}$.

- W.M. Schmidt (1970) : *For $m \geq 2$ let $L_0, \ldots, L_{m-1}$ be m independent linear forms in m variables with complex algebraic coefficients. Let $\epsilon > 0$. Then the set*

$$\{\mathbf{x} = (x_0, \ldots, x_{m-1}) \in \mathbf{Z}^m \; ; \; |L_0(\mathbf{x}) \cdots L_{m-1}(\mathbf{x})| \leq |\mathbf{x}|^{-\epsilon}\}$$

  is contained in the union of finitely many proper subspaces of $\mathbf{Q}^m$.
- Example : $m = 2$, $L_0(x_0, x_1) = x_0$, $L_1(x_0, x_1) = \alpha x_0 - x_1$. Roth's Theorem : *for any real algebraic number α, for any $\epsilon > 0$, the set of $p/q \in \mathbf{Q}$ with $|\alpha - p/q| < q^{-2-\epsilon}$ is finite.*

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Schmidt's subspace Theorem (several places)

W.M. Schmidt (1970) : *Let $m \geq 2$ be a positive integer, S a finite set of places of $\mathbf{Q}$ containing the infinite place. For each $v \in S$ let $L_{0,v}, \ldots, L_{m-1,v}$ be m independent linear forms in m variables with algebraic coefficients in the completion of $\mathbf{Q}$ at v. Let $\epsilon > 0$. Then the set of $\mathbf{x} = (x_0, \ldots, x_{m-1}) \in \mathbf{Z}^m$ such that*

$$\prod_{v \in S} |L_{0,v}(\mathbf{x}) \cdots L_{m-1,v}(\mathbf{x})|_v \leq |\mathbf{x}|^{-\epsilon}$$

is contained in the union of finitely many proper subspaces of $\mathbf{Q}^m$.

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Ridout's Theorem

- Ridout's Theorem : *for any real algebraic number* α, *for any* $\epsilon > 0$, *the set of* $p/q \in \mathbf{Q}$ *with* $q = 2^k$ *and* $|\alpha - p/q| < q^{-1-\epsilon}$ *is finite.*
- In Schmidt's Theorem take $m = 2$, $S = \{\infty, 2\}$,

  $L_{0,\infty}(x_0, x_1) = L_{0,2}(x_0, x_1) = x_0$,
  $L_{1,\infty}(x_0, x_1) = \alpha x_0 - x_1$, $\quad L_{1,2}(x_0, x_1) = x_1$.

  For $(x_0, x_1) = (q, p)$ with $q = 2^k$, we have

  $|L_{0,\infty}(x_0, x_1)|_\infty = q$, $\quad |L_{1,\infty}(x_0, x_1)|_\infty = |q\alpha - p|$,
  $|L_{0,2}(x_0, x_1)|_2 = q^{-1}$, $\quad |L_{1,2}(x_0, x_1)|_2 = |p|_2 \le 1$.

Further transcendence results

Consequences of Nesterenko 1996 result on the transcendence of values of theta series at rational points.

- The number $\displaystyle\sum_{n\ge 0} 2^{-n^2}$ is transcendental (D. Bertrand 1977 ; D. Duverney, K. Nishioka, K. Nishioka and I. Shiokawa 1998) (cf. Florian Luca lecture on tuesday).
- For the word

  $$\mathbf{u} = 012122122212222122222122222212222\ldots$$

  generated by the non–recurrent morphism $0 \mapsto 012$, $1 \mapsto 12$, $2 \mapsto 2$, the number $\eta = \displaystyle\sum_{k\ge 1} u_k 3^{-k}$ is transcendental.

Complexity of the continued fraction expansion of an algebraic number

- Similar questions arise by considering the continued fraction expansion of a real number instead of its g–ary expansion.
- Open question – A.Ya. Khintchine (1949) : *are the partial quotients of the continued fraction expansion of a non–quadratic irrational algebraic real number bounded ?*

Transcendence of continued fractions

- J. Liouville, 1844
- É. Maillet, 1906, O. Perron, 1929
- H. Davenport and K.F. Roth, 1955
- A. Baker, 1962
- J.L. Davison, 1989

Transcendence of continued fractions (continued)

- J.H. Evertse, 1996.
- M. Queffélec, 1998 : *transcendence of the Thue–Morse continued fraction.*
- P. Liardet and P. Stambul, 2000.
- J-P. Allouche, J.L. Davison, M Queffélec and L.Q. Zamboni, 2001 : *transcendence of Sturmian or morphic continued fractions.*
- C. Baxa, 2004.
- B. Adamczewski, Y. Bugeaud, J.L. Davison, 2005 : *transcendence of the Rudin-Shapiro and of the Baum–Sweet continued fractions.*

Transcendence of continued fractions

- Open question : *Do there exist algebraic numbers of degree at least three whose continued fraction expansion is generated by a morphism ?*
- B. Adamczewski, Y. Bugeaud (2004) : *The continued fraction expansion of an algebraic number of degree at least three cannot be generated by a binary morphism.*

Uniform rational approximation to a real number

Let $\xi \in \mathbf{R} \setminus \mathbf{Q}$.

- Dirichlet's box principle : for any real number $X \geq 1$, there exists $(x_0, x_1) \in \mathbf{Z}^2$ satisfying
$$0 < x_0 \leq X \quad \text{and} \quad |x_0\xi - x_1| \leq X^{-1}.$$
- Khintchine 1926 : there is no $\xi \in \mathbf{R}$ for which the exponent -1 can be lowered.
Gel'fond's transcendence criterion in 1948.
Refinements by H. Davenport and W.M. Schmidt in 1969.

Asymptotic rational approximation to a real number

Let $\lambda \geq 1$. Denote by S_λ the set of $\xi \in \mathbf{R}$ such that there are arbitrarily large values of X for which the system
$$0 < x_0 \leq X, \quad 0 < |x_0\xi - x_1| \leq X^{-\lambda}$$
has a solution $(x_0, x_1) \in \mathbf{R}^2$.
By Dirichlet $S_1 = \mathbf{R} \setminus \mathbf{Q}$.

- Liouville : the intersection for $\lambda > 1$ of all S_λ is not empty.
- A.Ya. Khintchine, K.F. Roth : for each $\lambda > 1$ the set S_λ has Lebesgue measure 0 and contains no algebraic number.
- (Cf. Igor Shparlinsky lecture on tuesday).

Uniform simultaneous approximation to ξ_1 and ξ_2

Let ξ_1 and ξ_2 be real numbers.

- Dirichlet's box principle : for any real number $X \geq 1$, there exists $(x_0, x_1, x_2) \in \mathbf{Z}^3$ satisfying

$$0 < x_0 \leq X, \quad |x_0\xi_1 - x_1| \leq \varphi(X), \quad |x_0\xi_2 - x_2| \leq \varphi(X)$$

  where $\varphi(X) = 1/[\sqrt{X}]$.
- If $1,\ \xi_1,\ \xi_2$ are linearly dependent over $\mathbf{Q}$, then the same is true with $\varphi(X) = c/X$ and $c = c(\xi_1, \xi_2) > 0$.

Asymptotic simultaneous approximation to ξ_1 and ξ_2

Let $\lambda \geq 1/2$. Denote by $S_\lambda^{(2)}$ the set of $(\xi_1, \xi_2) \in \mathbf{R}^2$ with $1, \xi_1, \xi_2$ $\mathbf{Q}$–linearly independent for which there are arbitrarily large values of X such that the system

$$0 < x_0 \leq X, \quad |x_0\xi_1 - x_1| \leq X^{-\lambda}, \quad |x_0\xi_2 - x_2| \leq X^{-\lambda}$$

has a solution $(x_0, x_1, x_2) \in \mathbf{Z}^3$.

- A.Ya. Khintchine (1926), J.W.S. Cassels (1957) : the intersection for $\lambda > 1/2$ of all $S_\lambda^{(2)}$ is not empty.
- W.M. Schmidt : for each $\lambda > 1/2$ the set $S_\lambda^{(2)}$ contains no element of $\overline{\mathbf{Q}}^2$.

Schmidt's subspace Theorem (again)

- W.M. Schmidt (1970) : *For $m \geq 2$ let $L_0, \ldots, L_{m-1}$ be m independent linear forms in m variables with algebraic coefficients. Let $\epsilon > 0$. Then the set*

$$\{\mathbf{x} = (x_0, \ldots, x_{m-1}) \in \mathbf{Z}^m \; ; \; |L_0(\mathbf{x}) \cdots L_{m-1}(\mathbf{x})| \leq |\mathbf{x}|^{-\epsilon}\}$$

is contained in the union of finitely many proper subspaces of $\mathbf{Q}^m$.

- Example : $m = 3$,

$$L_0(\mathbf{x}) = x_0, \; L_1(\mathbf{x}) = x_0\xi_1 - x_1, \; L_2(\mathbf{x}) = x_0\xi_2 - x_2.$$

For $\lambda > 1/2$, the set $S_\lambda^{(2)} \cap \overline{\mathbf{Q}}^2$ is empty.

Uniform simultaneous approximation to ξ and ξ^2

Let $\xi \in \mathbf{R}$. Take $\xi_1 = \xi$ and $\xi_2 = \xi^2$.

- Dirichlet's box principle : for any real number $X \geq 1$, there exists $(x_0, x_1, x_2) \in \mathbf{Z}^3$ satisfying

$$(*) \quad 0 < x_0 \leq X, \quad |x_0\xi - x_1| \leq \varphi(X), \quad |x_0\xi^2 - x_2| \leq \varphi(X)$$

where $\varphi(X) = 1/[\sqrt{X}]$.

- If ξ is algebraic of degree ≤ 2, the same is true with $\varphi(X) = c/X$ and $c = c(\xi) > 0$.

Uniform simultaneous approximation to a number and its square

Let $\xi \in \mathbf{R}$.

- For $\lambda > 1/2$, denote by E_λ the set of ξ which are not quadratic over $\mathbf{Q}$ and for which

$$(*) \quad 0 < x_0 \le X, \quad |x_0\xi - x_1| \le \varphi(X), \quad |x_0\xi^2 - x_2| \le \varphi(X)$$

  has a solution for any sufficiently large value of X with $\varphi(X) = X^{-\lambda}$
- Metrical result : the set E_λ has Lebesgue measure zero.
- Consequence of Schmidt's subspace Theorem : for $\lambda > 1/2$, the set E_λ contains no algebraic number.

Uniform approximation to ξ and ξ^2

- H. Davenport and W.M. Schmidt (1969) The set E_λ is empty for $\lambda > \Phi^{-1} = (-1+\sqrt{5})/2 = 0.618\ldots$.
- *Method of Davenport and Schmidt* : for three consecutive integers X, consider the solutions (x_0, x_1, x_2), (x_0', x_1', x_2'), (x_0'', x_1'', x_2'') to $(*)$ and show that the determinant

$$\begin{vmatrix} x_0 & x_1 & x_2 \\ x_0' & x_1' & x_2' \\ x_0'' & x_1'' & x_2'' \end{vmatrix}$$

  vanishes.

Result of D. Roy (2003)

The result of Davenport and Schmidt is optimal :
D. Roy (2003) produces examples of transcendental numbers ξ and constants $c > 0$ for which the inequalities

$$0 < x_0 \le X, \quad |x_0\xi - x_1| \le cX^{-\Phi^{-1}}, \quad |x_0\xi^2 - x_2| \le cX^{-\Phi^{-1}}$$

have a solution $(x_0, x_1, x_2) \in \mathbf{Z}^3$ for any sufficiently large value of X.

Billiards
Complexity of words
Words and transcendence
Words and Diophantine approximation

Diophantine approximation to the Fibonacci continued fraction

Let A and B be two distinct positive integers. Let $\xi \in (0,1)$ be the real number whose continued fraction expansion is obtained from the Fibonacci word w by replacing the letters a and b by A and B :

$$[0; A, B, A, A, B, A, B, A, A, B, A, A, B, A, B, A, A, \dots]$$

Then there exists $c > 0$ such that the inequalities

$$0 < x_0 \le X, \quad |x_0\xi - x_1| \le cX^{-\Phi^{-1}}, \quad |x_0\xi^2 - x_2| \le cX^{-\Phi^{-1}},$$

have a solution for any sufficiently large value of X (as above $\Phi^{-1} = (-1+\sqrt{5})/2 = 0.618\dots$).

Michel Waldschmidt http ://www.math.jussieu.fr/~miw/

More recent developments (2003–2006)

D. Roy : *approximation to real numbers by cubic algebraic integers.*
Y. Bugeaud and M. Laurent : *on exponents of Diophantine approximation and Sturmian continued fractions ; homogeneous and inhomogeneous Diophantine approximation.*
S. Fischler : *spectrum for the approximation of a real number and its square ; palindromic prefixes and Diophantine approximation ; palindromic prefixes and Episturmian words.*
M. Laurent : *exponents of Diophantine approximation in dimension two.*
Reference on Diophantine approximation : Y. Bugeaud, *Approximation by algebraic numbers*, Cambridge Tracts in Math. **160**, Cambridge Univ. Press 2004.

Multiple zeta values

Words occur also in the study of multiple zeta values.

This is another story. . .

Conference on Diophantine Analysis
and Related Fields 2006
in honor of Professor Iekata Shiokawa
Keio University Yokohama March 8, 2006

Congratulations to
Professor Iekata Shiokawa !