Tijdeman's mathematical contributions related to transcendental numbers

Michel Waldschmidt

Tijdeman, R.,
On a conjecture of Turán and Erdős,
Nederl. Akad. Wetensch. Proc. Ser. A **69**
$=$ Indag. Math. **28** (1966), 374–383.

List of references:

1. Jager, H., *A note on the vanishing of power sums,* (to be published in the Ann. Univ. Sci. Budapest.)

2. Sòs, Vérà T. and P. Turán, *On some new theorems in the theory of Diophantine approximations,* Acta Math. Acad. Sci. Hungar, **6**, 241–255 (1955).

3. Turán, P., *Eine neue Methode in der Analysis und deren Anwendungen*, Budapest, (1953)

4. Uchiyama, S., *Sur un problème posé par M. Turán,* Acta Arithm., **4**, 240–246 (1958)

5. Uchiyama, S., *Complex numbers with vanishing power sums,* Proc. Japan Acad., **33**, 10–12 (1957).

Tijdeman, R.,
On a conjecture of Turán and Erdős,
Nederl. Akad. Wetensch. Proc. Ser. A **69**
= Indag. Math. **28** (1966), 374–383.

Tijdeman, R.,
On the distribution of the values of certain functions,
Doctoral dissertation, Universiteit van Amsterdam, 1969, 94 pp.

Tijdeman, R.
On the number of zeros of general exponential polynomials,
Nederl. Akad. Wetensch. Proc. Ser. A **74**
= Indag. Math. **33** (1971), 1–7.

Tijdeman, R.
On the algebraic independence of certain numbers,
Nederl. Akad. Wetensch. Proc. Ser. A **74**
= Indag. Math. **33** (1971), 146–162.

Tijdeman, R.
Het handelsreizigersprobleem,
Stichting Math. Centr. en Inst. v. Actuar. en Econometrie,
S385 (SB3), 1967.

Tijdeman, R.
On a telephone problem,
Nieuw Arch. Wisk. (3) **19** (1971), 188–192.

Turán, Paul

Diophantine approximation and analysis.

Actes Congr. internat. Math. 1970, **1**, 519-528 (1971)

Let $a_1, \ldots, a_n$ be complex numbers. Consider the differential equation

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0 \tag{14}$$

and its characteristic equation

$$z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0. \tag{15}$$

Denote by M the maximal distance of the different roots of (15). Then the number of roots of any nonzero solution of (14) in a disk $|z| \leq R$ is at most

$$6n + 4RM. \tag{22}$$

Previous upper bounds depended upon the *minimal* distance m of the different roots of (15).

A.O. Gel'fond (1949, 1952)

K. Mahler (1967)

Previous upper bounds depended upon the *minimal* distance m of the different roots of (15).

A.O. Gel'fond (1949, 1952)

K. Mahler (1967)

A.J. van des Poorten,
Simultaneous algebraic approximation of functions,
Ph. D. Thesis, Univ. of New South Wales, Australia 1968.

Previous upper bounds depended upon the $minimal$ distance m of the different roots of (15).

A.O. Gel'fond (1949, 1952)

K. Mahler (1967)

A.J. van des Poorten,
Simultaneous algebraic approximation of functions,
Ph. D. Thesis, Univ. of New South Wales, Australia 1968.

Turán, Paul,
Eine neue Methode in der Analysis und deren Anwendungen,
Akadémiai Kiadó, Budapest, 1953. 196 pp.

Quoted from p. 523 of Paul Turán's lecture at the International Congress in Nice (1970):

Recently however R. Tydeman discovered that again a proper use of (6) leads to the much more elegant upper bound (22) which has also the advantage over our upper bound that it does not depend on m too. A consequence of (22) (i.e. a propagation of the effect of (6)) to the theory of transcendental numbers was noted by J. Coates. This refers to the fundamental Theorem I of Gel'fond dealing with algebraic independence over the rationals of certain types of numbers (see p. 132 of his book "Transcendental and algebraic numbers") where the most inconvenient restriction (112) from the hypotheses of this theorem can simply be dropped.

Gel'fond, A. O.,
Transcendental and algebraic numbers,
Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1952.
Translated from the first Russian edition by Leo F. Boron,
Dover Publications, Inc., New York 1960.

We extend the rational number field by adjoining one transcendental number to it, and we extend the field R_0 *thus obtained by adjoining to it in turn a root of an algebraic equation with coefficients in the field* R_0. *Any such field or, simply, a finite algebraic field will be called a field* R_1.

Field R_1 = field of transcendence degree ≤ 1 over $\mathbf{Q}$.

Theorem I. (Gel'fond, 1952). *Suppose the numbers* η_0, η_1, η_2, *as well as the numbers* $1, \alpha_1, \alpha_2$ *are linearly independent over the rational field and that the inequality*

$$(112) \quad |x_0\eta_0 + x_1\eta_1 + x_2\eta_2| > e^{-\tau x \ln x},\ |x_i| \leq x,\ i = 0, 1, 2$$

holds, where $\tau > 0$ *is some constant, and* $x_0, x_1, x_2, \sum_1^3 |x_i| > 0$ *are rational integers, for* $x > x'$. *Then no extension of the rational number field by means of adjunction to it of the* 11 *numbers*

$$(113) \quad \alpha_1,\ \alpha_2,\ e^{\eta_i \alpha_k}, \quad i = 0, 1, 2, \quad k = 0, 1, 2, \quad \alpha_0 = 1,$$

will yield the field R_1.

Theorem I. (improved by R. Tijdeman). *Suppose the numbers* η_0, η_1, η_2*, as well as the numbers* $1, \alpha_1, \alpha_2$ *are linearly independent over the rational field.*

Then no extension of the rational number field by means of adjunction to it of the 11 *numbers*

$$(113) \qquad \alpha_1,\ \alpha_2,\ e^{\eta_i \alpha_k}, \quad i = 0, 1, 2, \quad k = 0, 1, 2, \quad \alpha_0 = 1,$$

will yield the field R_1*.*

Tijdeman, R.
On the algebraic independence of certain numbers,
Nederl. Akad. Wetensch. Proc. Ser. A **74**
= Indag. Math. **33** (1971), 146–162.

Example. Let c be a quadratic irrational and b an algebraic number in such a manner that $1, b, c, bc$ are linearly independent over $\mathbf{Q}$. Let $a \neq 0, 1$. Then at least two of the numbers

$$a, a^b, a^c, a^{bc}$$

are algebraically independent over the rational number field.

Theorem (R. Tijdeman, 1971). *Suppose the numbers* $\alpha_0, \alpha_1, \ldots, \alpha_m$ *as well as the numbers* $\eta_0, \eta_1, \ldots, \eta_\ell$ *are linearly independent over* $\mathbf{Q}$. *Then the extension of the rational field by means of adjunction to it of the following numbers will have transcendence degree* ≥ 2*:*

1. *For* $m = \ell = 2$: $\qquad \alpha_k,\ e^{\alpha_k \eta_j}, \quad k, j = 0, 1, 2.$

2. *For* $m = 2,\ \ell = 1$:
$$\alpha_k,\ \eta_j,\ e^{\alpha_k \eta_j}, \quad k = 0, 1, 2 \ \text{ and } \ j = 0, 1.$$

3. *For* $m = 3,\ \ell = 1$:
$$\alpha_k,\ e^{\alpha_k \eta_j}, \quad k = 0, 1, 2, 3 \ \text{ and } \ j = 0, 1.$$

Spira, R.,
A lemma in transcendental number theory
Trans. Am. Math. Soc. **146**, 457-464 (1969).

Spira, R.,
A lemma in transcendental number theory
Trans. Am. Math. Soc. **146**, 457-464 (1969).

Šmelev, A. A.,
The algebraic independence of certain transcendental numbers,
Mat. Zametki **3**, (1968), 51–58.

Review by R. Spira in Mathematical Reviews `M.R.385721`

38 5721 10.32

Šmelev, A. A.

The algebraic independence of certain transcendental numbers. (Russian)

Mat. Zametki **3** 1968 51–58

Let α_1 and α_2 be algebraic numbers, the logarithms of which are linearly independent over the rational field, and let β be a quadratic algebraic number. Using unproved lemmas (see below), the author presents a proof that the numbers ${\alpha_1}^\beta, {\alpha_2}^\beta$, $(\ln\alpha_2)/\ln\alpha_1$ cannot be algebraically expressed in terms of one of them. The reviewer did not work through the details.

The lemmas in question appear in A. O. Gel′fond's book [*Transcendental and algebraic numbers* (Russian), GITTL, Moscow, 1952; MR **15**, 292; English translation, Dover, New York, 1960; MR **22** #2598]. The present reviewer considers Lemmas I and IV of Chapter I, § 2, and Lemmas I-III and V-VII of Chapter III, § 5.

The phrase translated as "all different from zero" should be replaced everywhere by "not all zero".

In Chapter I, the inequalities in (19) and on the following line are not strict. The right hand side of (26) should be ${H_1}^m/(2mn_1+1)$ with corresponding changes in the proof. In Lemma IV, f_1 and f_2 can be assumed to be polynomials of non-negative degrees with complex coefficients. Inequality (30) is not strict if f_1 and f_2 are constant.

In Chapter III, Lemmas II and II′ refer to polynomials with complex coefficients. In equation (126), "cos" should be "sin". In equation (139) the factor h^m should occur in front of the product sign. There is similar poor notation involving product signs throughout this section. Different integration variables are needed in the display after (139).

Lemma III is very obscure and not proved. The reviewer in a forthcoming article (to appear in Trans. Amer. Math. Soc.) replaces this lemma with a simpler lemma which will have the same force.

In Lemma V, the sentence "In case $\nu = 1$, the number $a = 1$" is part of the definition of the quantity a. In inequalities (166) and (168) the third "α" should be "a". The words "not greater than" should be omitted, as well as the first paragraph of the proof. In Lemma VI, the inequality $H > H'(\alpha)$ should be added to the conditions (169). The second inequality of (179) is not necessarily strict.

In Lemma VII, the inequalities for $|Q_q(\alpha)|$ in (181) and (183) appear to be falsely derived, and the reviewer sees no way of repairing the proof. The proofs of Theorems I and II in § 5 depend on this lemma, as does the result of the present author. *R. Spira*

Let α_1 and α_2 be algebraic numbers, the logarithms of which are linearly independent over the rational field, and let β be a quadratic algebraic number. **Using unproved lemmas (see below), the author presents a proof** that the numbers ${\alpha_1}^{\beta}, {\alpha_2}^{\beta}$, $(\ln \alpha_2)/\ln \alpha_1$ cannot be algebraically expressed in terms of one of them. The reviewer did not work through the details.

The lemmas in question appear in A. O. Gel'fond's book [Transcendental and algebraic numbers (Russian), GITTL, Moscow, 1952; MR 15, 292; English translation, Dover, New York, 1960; MR 22 #2598]. The present reviewer considers Lemmas I and IV of Chapter I, §2, and Lemmas I-III and V-VII of Chapter III, §5.

The phrase translated as "all different from zero" should be replaced everywhere by "not all zero".

In Chapter I, the inequalities in (19) and on the following line are not strict. The right hand side of (26) should be ${H_1}^m/(2mn_1+1)$ with corresponding changes in the proof. In Lemma IV, f_1 and f_2 can be assumed to be polynomials of non-negative degrees with complex coefficients. Inequality (30) is not strict if f_1 and f_2 are constant.

In Chapter III, Lemmas II and III refer to polynomials with complex coefficients. In equation (126), "cos" should be "sin". In equation (139) the factor h^m should occur in front of the product sign. There is similar poor notation involving product signs throughout this section. Different integration variables are needed in the display after (139).

Lemma III is very obscure and not proved. The reviewer in a forthcoming article (to appear in Trans. Amer. Math. Soc.) replaces this lemma with a simpler lemma which will have the same force.

In Lemma V, the sentence "In case $\nu = 1$, the number $a = 1$" is part of the definition of the quantity a. In inequalities (166) and (168) the third "α" should be "a". The words "not greater than" should be omitted, as well as the first paragraph of the proof. In Lemma VI, the inequality $H > H'(\alpha)$ should be added to the conditions (169). The second inequality of (179) is not necessarily strict.

In Lemma VII, the inequalities for $|Q_q(\alpha)|$ in (181) and (183) appear to be falsely derived, and the reviewer sees no way of repairing the proof. The proofs of Theorems I and II in §5 depend on this lemma, as does the result of the present author.

R. Spira

---

41 5307 10.32

Spira, Robert

A lemma in transcendental number theory.

Trans. Amer. Math. Soc. **146** 1969 457–464

This paper concerns a lemma of A. O. Gel′fond, which plays an important part in the theory of algebraic independence of numbers [see Gel′fond, *Transcendental and algebraic numbers*, pp. 140–141, Lemma III, Dover, New York, 1960; MR **22** #2598]. As was, however, pointed out by N. I. Fel′dman in a letter to the editor of the Trans. Amer. Math. Soc., the paper contains a mistake, which cannot easily be repaired, viz. on p. 462, where a false integral representation for $B_{i,n}$ is given. Although there might be some justification in the author's objections against Gel′fond's formulation and proof of the lemma, the reviewers, with Fel′dman, think the criticism of the author on this beautiful work of Gel′fond is far too severe. But the author's idea that Gel′fond's lemma can be sharpened is quite correct. This appears from a result by the second reviewer, which improves the author's assertion and will appear in Nederl. Akad. Wetensch. Proc. Ser. A. *J. Popken*

This paper concerns a lemma of A. O. Ge'fond, which plays an important part in the theory of algebraic independence of numbers [see Ge'fond, Transcendental and algebraic numbers, pp. 140–141, Lemma III, Dover, New York, 1960; MR **22** #2598]. **As was, however, pointed out by N. I. Fel'dman in a letter to the editor of the Trans. Amer. Math. Soc., the paper contains a mistake, which cannot easily be repaired**, viz. on p. 462, where a false integral representation for $B_{i,n}$ is given. Although there might be some justification in the author's objections against Ge'fond's formulation and proof of the lemma, the reviewers, with Fel'dman, think the criticism of the author on this beautiful work of Ge'fond is far too severe. But the author's idea that Ge'fond's lemma can be sharpened is quite correct. This appears from **a result by the second reviewer**, which improves the author's assertion and will appear in Nederl. Akad. Wetensch. Proc. Ser. A. *J. Popken*

Tijdeman, R.
On the algebraic independence of certain numbers,
Nederl. Akad. Wetensch. Proc. Ser. A **74**
= Indag. Math. **33** (1971), 146–162.

Lemma 6. Let $\alpha \in \mathbf{C}$. If, for every rational integer $N \geq N_0$, there exists a non-trivial $P \in \mathbf{Z}[X]$ such that

$$\log|P(\alpha)| \leq -7N^2, \quad n(P) \leq N, \quad \log H(P) \leq N,$$

then α is an algebraic number.

Here, $n(P)$ is the degree of P and $H(P)$ its height, maximum absolute value of its coefficients.

Large transcendence degree

Criteria for algebraic independence due to Brownawell, Chudnovskĭi, Nesterenko, Philippon,. . .

Open problem: remove the technical hypothesis.

Known only for small transcendence degree.

Large transcendence degree

Criteria for algebraic independence due to
Brownawell, Chudnovskiĭ, Nesterenko, Philippon,. . .

Open problem: remove the technical hypothesis.

Known only for small transcendence degree.

Conjecture of S. Lang -
disproved by earlier counterexample of J.W.S. Cassels.

Large transcendence degree

Criteria for algebraic independence due to Brownawell, Chudnovskĭi, Nesterenko, Philippon,. . .

Open problem: remove the technical hypothesis.

Known only for small transcendence degree.

Help, Rob!

Further results on exponential polynomials:

Balkema, A. A.; Tijdeman, R.
Some estimates in the theory of exponential sums,
Acta Math. Acad. Sci. Hungar. **24** (1973), 115–133.

For E a solution of a linear differential equation of order n, denoting by M the maximum of the absolute value of the roots of the associated characteristic equation, for any $\gamma > 1$ and $R > 1$,

$$|E|_{\gamma R} \leq \frac{\gamma^n - 1}{\gamma - 1} e^{RM(\gamma+1)} |E|_R.$$

Voorhoeve, M.; van der Poorten, A.J.; Tijdeman, R.,
On the number of zeros of certain functions,
Nederl. Akad. Wet., Proc., Ser. A **78**, 407-416 (1975)

Voorhoeve, Marc,
Zeros of exponential polynomials.
Rijksuniversiteit te Leiden. 72 p. (1977).

Gel'fond, A. O.; Linnik, Y. V.,
Elementary methods in the analytic theory of numbers.
Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow 1962.

The number of real *zeros of a nonzero* real *exponential polynomial*

$$E(z) = \sum_{k=1}^{\ell} \sum_{j=1}^{p_k} b_{kj} z^{j-1} e^{w_k z}$$

is at most

$$n = \sum_{k=1}^{\ell} p_k$$

Gel'fond, A. O.; Linnik, Y. V.,
Elementary methods in the analytic theory of numbers.
Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow 1962.

The number of real *zeros of a nonzero* real *exponential polynomial*

$$E(z) = \sum_{k=1}^{\ell} \sum_{j=1}^{p_k} b_{kj} z^{j-1} e^{w_k z}$$

is at most

$$n = \sum_{k=1}^{\ell} p_k$$

Proof: `use Rolle's Theorem and induction.`

Tijdeman, R.,
An auxiliary result in the theory of transcendental numbers,
J. Number Theory **5** (1973), 80–94.

Cijsouw, P. L.; Tijdeman, R.,
An auxiliary result in the theory of transcendental numbers. II.
Duke Math. J. **42** (1975), 239–247.

Assume

$$E(z)=\sum_{k=1}^{\ell}\sum_{j=1}^{p_k} b_{kj} z^{j-1} e^{w_k z}$$

with complex numbers b_{kj} and pairwise distinct complex numbers w_k. Let $\beta_0,\dots,\beta_{s-1}\in\mathbf{C}$ be pairwise distinct. Let $M\geq|w_k|$, $R\geq|\beta_\sigma|$; set $n=p_1+\cdots+p_\ell$ and $N=r_0+\cdots+r_{s-1}$.

Assume

$$E(z) = \sum_{k=1}^{\ell} \sum_{j=1}^{p_k} b_{kj} z^{j-1} e^{w_k z}$$

with complex numbers b_{kj} and pairwise distinct complex numbers w_k. Let $\beta_0, \ldots, \beta_{s-1} \in \mathbf{C}$ be pairwise distinct. Let $M \geq |w_k|$, $R \geq |\beta_\sigma|$; set $n = p_1 + \cdots + p_\ell$ and $N = r_0 + \cdots + r_{s-1}$.

Zero estimate: If $N > 3(n-1) + 4RM$, then the conditions

$$E^{(\varrho)}(\beta_\sigma) = 0 \quad \text{for} \quad 0 \leq \varrho \leq r_\sigma - 1, \quad 0 \leq \sigma \leq s-1$$

imply

$$b_{kj} = 0 \quad \text{for} \quad 1 \leq j \leq p_k, \quad 1 \leq k \leq \ell.$$

Assume

$$E(z) = \sum_{k=1}^{\ell} \sum_{j=1}^{p_k} b_{kj} z^{j-1} e^{w_k z}$$

with complex numbers b_{kj} and pairwise distinct complex numbers w_k. Let $\beta_0, \ldots, \beta_{s-1} \in \mathbf{C}$ be pairwise distinct. Let $M \geq |w_k|$, $R \geq |\beta_\sigma|$; set $n = p_1 + \cdots + p_\ell$ and $N = r_0 + \cdots + r_{s-1}$.

Small Values Lemma: If $N > 2n + 13RM$, then

$$\max_{\substack{1 \leq j \leq p_k \\ 1 \leq k \leq \ell}} |b_{kj}| \leq C \max_{\substack{0 \leq \varrho \leq r_\sigma - 1 \\ 0 \leq \sigma \leq s-1}} |E^{(\varrho)}(\beta_\sigma)|$$

with an explicit value for C.

Further developments:

Brownawell, Masser, Wüstholz, Nesterenko, Philippon,. . .

Cijsouw, Pieter Leendert,
Transcendence measures,
Doctoral dissertation, Universiteit van Amsterdam, 1972, 107 pp.

Cijsouw, Pieter Leendert,
Transcendence measures,
Doctoral dissertation, Universiteit van Amsterdam, 1972, 107 pp.

How to measure the transcendence measure?

Cijsouw, Pieter Leendert,
Transcendence measures,
Doctoral dissertation, Universiteit van Amsterdam, 1972, 107 pp.

Linear forms and simultaneous approximations

Cijsouw, Pieter Leendert,
Transcendence measures,
Doctoral dissertation, Universiteit van Amsterdam, 1972, 107 pp.

Linear forms and simultaneous approximations

Bijlsma, A.,
Simultaneous approximations in transcendental number theory,
Mathematical Centre Tracts, **94**.
Mathematisch Centrum, Amsterdam, 1978.

On the simultaneous approximation of a, b and a^b.

Further contributions to transcendence

Cijsouw, P. L.; Tijdeman, R.,
On the transcendence of certain power series of algebraic numbers,
Acta Arith. **23** (1973), 301–305.

Adhikari, S. D.; Saradha, N.; Shorey, T. N.; Tijdeman, R.
Transcendental infinite sums,
Indag. Math. (N.S.) **12** (2001), no. 1, 1–14.

Saradha, N.; Tijdeman, R.
On the transcendence of infinite sums of values of rational functions,
J. London Math. Soc. (2) **67** (2003), no. 3, 580–592.

Arithmetic nature of

$$\sum_{\substack{n\geq 0\\ Q(n)\neq 0}} \frac{P(n)}{Q(n)}.$$

Also series involving the Fibonacci numbers F_n.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1,$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{4n+1} - \frac{3}{4n+2} + \frac{1}{4n+3} + \frac{1}{4n+4} \right) = 0$$

are rational numbers

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+2)} = \log 2,$$

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)(2n+1)(4n+1)} = \frac{\pi}{3},$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1} = \frac{1}{2} + \frac{\pi}{2} \cdot \frac{e^{\pi}+e^{-\pi}}{e^{\pi}-e^{-\pi}}$$

are transcendental numbers.

The number

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2}} = 1$$

is rational, while

$$\sum_{n=0}^{\infty} \frac{1}{F_{2^n}} = \frac{7-\sqrt{5}}{2}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+1}} = \frac{1-\sqrt{5}}{2},$$

and

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}+1} = \frac{\sqrt{5}}{2},$$

are algebraic irrational numbers.

Each of the numbers

$$\sum_{n=1}^{\infty} \frac{1}{F_n}, \quad \sum_{n=1}^{\infty} \frac{1}{F_n + F_{n+2}}, \quad \sum_{n \geq 1} \frac{1}{F_1 F_2 \cdots F_n}$$

is irrational, but it is not known whether they are algebraic or transcendental.

The numbers

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}}, \quad \sum_{n=1}^{\infty} \frac{1}{F_n^2}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n^2}, \quad \sum_{n=1}^{\infty} \frac{n}{F_{2n}},$$

are all transcendental.

S.D. Adhikari, N. Saradha and T.N. Shorey and R. Tijdeman show that the numbers

$$\sum_{n=0}^{\infty} \frac{1}{(3n+1)(3n+2)(3n+3)}, \qquad \sum_{n=1}^{\infty} \frac{\chi(n)}{n}, \qquad \sum_{n=1}^{\infty} \frac{F_n}{n2^n},$$

where χ is any non-principal Dirichlet character, are transcendental and give approximation measures for them.

The proofs depend on Baker's theory on linear forms in logarithms.

Hančl, J.; Tijdeman, R.,
On the irrationality of Cantor series,
J. reine angew. Math., to appear.

Tijdeman, R.; Pingzhi, Yuan,
On the rationality of Cantor and Ahmes series,
Indag. Math. (N.S.) **13** (2002), no.3, 407–418.

$$\sum_{n=1}^{\infty} \frac{b_n}{a_1 \cdots a_n}, \quad \sum_{n=1}^{\infty} \frac{1}{a_n}, \quad \sum_{n=1}^{\infty} \frac{b_n}{a_n}.$$

Catalan's conjecture

Tijdeman, R.,
On the equation of Catalan,
Acta Arith. **29** (1976), no. 2, 197–209.

Tijdeman, R.,
Exponential Diophantine equations,
Proceedings of the International Congress of Mathematicians (Helsinki, 1978), pp. 381–387,
Acad. Sci. Fennica, Helsinki, 1980.

Perfect powers

Squares:

$$1, \quad 4, \quad 9, \quad 16, \quad 25, \quad 36, \quad 49, \quad 64, \quad 81, \quad 100, \quad 121,$$
$$144, \quad 169, \quad 196, \quad 225, \quad 256, \quad 289, \quad 324, \quad 361, \ldots$$

Perfect powers

Squares:

$$1,\quad 4,\quad 9,\quad 16,\quad 25,\quad 36,\quad 49,\quad 64,\quad 81,\quad 100,\quad 121,$$
$$144,\quad 169,\quad 196,\quad 225,\quad 256,\quad 289,\quad 324,\quad 361,\ldots$$

Cubes:

$$1,\quad 8,\quad 27,\quad 64,\quad 125,\quad 216,\quad 343,\quad 512,\quad 729,\ldots$$

Perfect powers

Squares:

$$1, \quad 4, \quad 9, \quad 16, \quad 25, \quad 36, \quad 49, \quad 64, \quad 81, \quad 100, \quad 121,$$
$$144, \quad 169, \quad 196, \quad 225, \quad 256, \quad 289, \quad 324, \quad 361, \ldots$$

Cubes:

$$1, \quad 8, \quad 27, \quad 64, \quad 125, \quad 216, \quad 343, \quad 512, \quad 729, \ldots$$

Fifth powers:

$$1, \quad 32, \quad 243, \quad 1\,024, \quad 3\,125, \quad 7\,776, \quad 16\,807, \ldots$$

Perfect powers

Squares:

$$1, \quad 4, \quad 9, \quad 16, \quad 25, \quad 36, \quad 49, \quad 64, \quad 81, \quad 100, \quad 121,$$
$$144, \quad 169, \quad 196, \quad 225, \quad 256, \quad 289, \quad 324, \quad 361, \ldots$$

Cubes:

$$1, \quad 8, \quad 27, \quad 64, \quad 125, \quad 216, \quad 343, \quad 512, \quad 729, \ldots$$

Fifth powers:

$$1, \quad 32, \quad 243, \quad 1\,024, \quad 3\,125, \quad 7\,776, \quad 16\,807, \ldots$$

Seventh powers:

$$1, \quad 128, \quad 2\,187, \quad 16\,384, \quad 78\,125, \quad 279\,936, \ldots$$

The sequence of perfect powers

$1,\ 4,\ 8,\ 9,\ 16,\ 25,\ 27,\ 32,\ 36,\ 49,\ 64,\ 81,\ 100,\ 121,\ 125,\ 128, \ldots$

The sequence of perfect powers

$1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, 128, \ldots$

Catalan's Conjecture:

The only consecutive numbers in this sequence are 8 *and* 9*.*

The sequence of perfect powers

$1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, 128, \ldots$

Catalan's Conjecture:

The only consecutive numbers in this sequence are 8 and 9.

The sequence of differences

$3,\ 4,\ 1,\ 7,\ 9,\ 2,\ 5,\ 4,\ 13,\ 15,\ 17,\ 19,\ 21,\ 4,\ 3, \ldots$

between successive numbers in the sequence of perfect powers

The sequence of perfect powers

$1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, 128, \ldots$

Catalan's Conjecture:

The only consecutive numbers in this sequence are 8 *and* 9.

Catalan's Conjecture:

In the sequence of differences

$3,\ 4,\ 1,\ 7,\ 9,\ 2,\ 5,\ 4,\ 13,\ 15,\ 17,\ 19,\ 21,\ 4,\ 3, \ldots$

between successive numbers in the sequence of perfect powers 1 *appears only once.*

The sequence of perfect powers

$1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, 128, \ldots$

Catalan's Conjecture:

The only consecutive numbers in this sequence are 8 *and* 9.

Pillai's Conjecture:

The sequence of differences

$3, 4, 1, 7, 9, 2, 5, 4, 13, 15, 17, 19, 21, 4, 3, \ldots$

between successive numbers in the sequence of perfect powers tends to infinity.

The sequence of perfect powers

$1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, 128, \ldots$

Catalan's Conjecture:

The only consecutive numbers in this sequence are 8 *and* 9*.*

Pillai's Conjecture:

The sequence of differences

$$3,\ 4,\ 1,\ 7,\ 9,\ 2,\ 5,\ 4,\ 13,\ 15,\ 17,\ 19,\ 21,\ 4,\ 3, \ldots$$

between successive numbers in the sequence of perfect powers tends to infinity.

Means: *For any* $k \geq 1$*, the equation* $x^p - y^q = k$ *in rational integers* p*,* q*,* x*,* y*, all of which are* > 1*, has only finitely many solutions.*

Theorem (Tijdeman, 1976) Catalan's equation $x^p - y^q = 1$ has only finitely many solutions in integers $p > 1$, $q > 1$, $x > 1$, $y > 1$. Effective bounds for the solutions p, q, x, y can be given.

This is the first and only one case so far (namely $k = 1$) where an answer to Pillai's conjecture on the equation $x^p - y^q = k$ is known.

Refinement of a result of A. Baker:

Theorem (Tijdeman, 1976) *Let* $a_1, \ldots, a_n$ *be non zero algebraic numbers with degrees at most* d *and let the heights of* $a_1, \ldots, a_{n-1}$ *and* a_n *be at most* A' *(*≥ 2*) and* A *(*≥ 2*) respectively. Then there exists an effectively computable constant* $C = C(d, n)$ *such that the inequalities*

$$0 < |b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n| < \exp\bigl(-C(\log A')^{2n^2+16n} \log A \log B\bigr)$$

have no solution in rational integers $b_1, \ldots, b_n$ *with absolute values at most* B *(*≥ 2*).*

Improvement of a lemma on binomial polynomials

Define

$$\Delta(x;k) = \frac{(x+1)\cdots(x+k)}{k!}$$

with $\Delta(x;0) = 1$. For any integers $\ell \geq 0$, $m \geq 0$, denote

$$\Delta(x;k,\ell,m) = \frac{1}{m!}\left(\frac{d}{dx}\right)^m \left(\Delta(x;k)\right)^\ell.$$

We signify by $\nu(k)$ the least common multiple of $1,\ldots,k$.

Lemma. *Let q and qx be positive integers. Then*

$$q^{2k\ell}\big(\nu(k)\big)^m\Delta(x;k,\ell,m)$$

is a positive integer and we have

$$\Delta(x;k,\ell,m)\le 4^{\ell(x+k)},\quad \nu(k)\le 4^k.$$

Catalan's conjecture

(end of the story)

Complete answer by Preda Mihailescu in 2002.

See

Yu.F. Bilu,
Catalan's conjecture (after Mihailescu),
Séminaire Bourbaki, Exposé 909, 55ème année (2002-2003)

`http://www.math.u-bordeaux.fr/~yuri/`

Also a recent lecture at Oberwolfach.

Conjecture (Pillai). For given positive integers a, b and k, the equation

$$ax^p - by^q = k$$

has only finitely many solutions in positive integers p, q, x and y with $p > 1$, $q > 1$, $x > 1$, $y > 1$ and $(p, q) \neq (2, 2)$.

Oesterlé-Masser abc Conjecture

Stewart, C. L.; Tijdeman, R.,
On the Oesterlé-Masser conjecture,
Monatsh. Math. **102** (1986), no. 3, 251–257.

Oesterlé-Masser abc Conjecture

Stewart, C. L.; Tijdeman, R.,
On the Oesterlé-Masser conjecture,
Monatsh. Math. **102** (1986), no. 3, 251–257.

If a, b, c *are relatively prime positive integers with* $a + b = c$, *then*

$$\log c \le \kappa R^{15} \quad \textit{with} \quad R = R(abc) = \prod_{p \mid abc} p.$$

Oesterlé-Masser abc Conjecture

Stewart, C. L.; Tijdeman, R.,
On the Oesterlé-Masser conjecture,
Monatsh. Math. **102** (1986), no. 3, 251–257.

If a, b, c *are relatively prime positive integers with* $a+b=c$, *then*

$$\log c \le \kappa R^{15} \quad \textit{with} \quad R = R(abc) = \prod_{p|abc} p.$$

Later refinement by Stewart and Yu Kunrui:

$$\log c \le \kappa R^{1/3} (\log R)^3.$$

Oesterlé-Masser abc Conjecture

Stewart, C. L.; Tijdeman, R.,
On the Oesterlé-Masser conjecture,
Monatsh. Math. **102** (1986), no. 3, 251–257.

If a, b, c *are relatively prime positive integers with* $a + b = c$, *then*

$$\log c \leq \kappa R^{15} \quad \textit{with} \quad R = R(abc) = \prod_{p|abc} p.$$

abc Conjecture:

$$c \leq \kappa(\epsilon) R^{1+\epsilon}.$$

Stewart, C. L.; Tijdeman, R.,
On the Oesterlé-Masser conjecture,
Monatsh. Math. **102** (1986), no. 3, 251–257.

Let $\delta > 0$. Then there exists infinitely many triple of positive integers (a, b, c) such that $a + b = c$, $\gcd(a, b, c) = 1$ and

$$c > R \exp\left((4-\delta)\frac{\sqrt{\log R}}{\log\log R}\right).$$

Diophantine Equations

Shorey, T. N.; Tijdeman, R.,
Exponential Diophantine equations,
Cambridge Tracts in Mathematics, **87**.
Cambridge University Press, Cambridge, 1986.

Joint works with F. Beukers, P. Erdős, J.H. Evertse, K. Győry, R.J. Kooman, M. Mignotte, P. Moree, K. Ramachandra, N. Saradha, A. Schinzel, T.N. Shorey, C.L. Stewart, G.N ten Have, Wang Lian Xiang,. . .

Happy Birthday Rob!