Seventh lecture: April 21, 2011

7.2 Chen's integrals

Chen iterated integrals are defined by induction as follows. Let $\varphi_1, \dots, \varphi_p$ be holomorphic differential forms on a simply connected open subset D of the complex plane and let x and y be two elements in D. Define, as usual, $\int_x^y \varphi_1$ as the value, at y, of the primitive of φ_1 which vanishes at x. Next, by induction on p, define

$$\int_x^y \varphi_1 \cdots \varphi_p = \int_x^y \varphi_1(t) \int_x^t \varphi_2 \cdots \varphi_p.$$

By means of a change of variables

$$t \longmapsto x + t(y-x)$$

one can assume that $x = 0$, $y = 1$ and that D contains the real segment $[0,1]$. In this case the integral is

$$\int_0^1 \varphi_1 \cdots \varphi_p = \int_{\Delta_p} \varphi_1(t_1)\varphi_2(t_2)\cdots\varphi_p(t_p),$$

where the domain of integration Δ_p is the simplex of $\mathbf{R}^p$ defined by

$$\Delta_p = \bigl\{(t_1, \dots, t_p) \in \mathbf{R}^p \ ,\ 1 > t_1 > \cdots > t_p > 0\bigr\}.$$

In our applications the open set D will be the open disk $|z - (1/2)| < 1/2$, the differential forms will be dt/t and $dt/(1-t)$, so one needs to take care of the fact that the limit points 0 and 1 of the integrals are not in D. One way is to integrate for ϵ_1 to $1-\epsilon_2$ and to let ϵ_1 and ϵ_2 tend to 0. Here we just ignore these convergence questions by restricting our discussion to the convergent words and to the algebra $\mathfrak{H}^0$ they generate.

The product of two integrals is a Chen integral, and more generally the product of two Chen integrals is a Chen integral. This is where the shuffle comes in. We consider a special case: the product of the two integrals

$$\int_{1>t_1>t_2>0} \varphi_1(t_1)\varphi_2(t_2) \int_{1>t_3>0} \varphi_3(t_3)$$

is the sum of three integrals

$$\int_{1>t_1>t_2>t_3>0} \varphi_1(t_1)\varphi_2(t_2)\varphi_3(t_3),$$

$$\int_{1>t_1>t_3>t_2>0} \varphi_1(t_1)\varphi_3(t_3)\varphi_2(t_2)$$

and

$$\int_{1>t_3>t_1>t_2>0} \varphi_3(t_3)\varphi_1(t_1)\varphi_2(t_2).$$

Consider for instance the third integral: we write it as

$$\int_{1>t_{\tau(1)}>t_{\tau(2)}>t_{\tau(3)}>0} \varphi_{\tau(1)}(t_{\tau(1)})\varphi_{\tau(2)}(t_{\tau(2)})\varphi_{\tau(3)}(t_{\tau(3)}).$$

The permutation τ of $\{1,2,3\}$ is $\tau(1)=3$, $\tau(2)=1$, $\tau(3)=2$, and it is one of the three permutations of $\mathfrak{S}_3$ which is of the form σ^{-1} where $\sigma(1)<\sigma(2)$.

The shuffle ш will be defined in §7.3 so that the next lemma holds:

Lemma 29. *Let $\varphi_1,\dots,\varphi_{p+q}$ be differential forms with $p\ge 0$ and $q\ge 0$. Then*

$$\int_0^1 \varphi_1\cdots\varphi_p \int_0^1 \varphi_{p+1}\cdots\varphi_{p+q} = \int_0^1 \varphi_1\cdots\varphi_p ш \varphi_{p+1}\cdots\varphi_{p+q}.$$

Proof. Define $\Delta'_{p,q}$ as the subset of $\Delta_p\times\Delta_q$ of those elements $(z_1,\dots,z_{p+q})$ for which we have $z_i\neq z_j$ for $1\le i\le p<j\le p+q$. Hence

$$\int_0^1 \varphi_1\cdots\varphi_p \int_0^1 \varphi_{p+1}\cdots\varphi_{p+q} = \int_{\Delta_p\times\Delta_q} \varphi_1\cdots\varphi_{p+q} = \int_{\Delta'_{p,q}} \varphi_1\cdots\varphi_{p+q}$$

where $\Delta'_{p,q}$ is the disjoint union of the subsets $\Delta^{\sigma}_{p,q}$ defined by

$$\Delta^{\sigma}_{p,q} = \bigl\{(t_1,\dots,t_{p+q})\;;\;1>t_{\sigma^{-1}(1)}>\cdots>t_{\sigma^{-1}(p+q)}>0\bigr\},$$

for σ running over the set $\mathfrak{S}_{p,q}$ of permutations of $\{1,\dots,p+q\}$ satisfying

$$\sigma(1)<\sigma(2)<\cdots<\sigma(p) \quad\text{and}\quad \sigma(p+1)<\sigma(p+2)<\cdots<\sigma(p+q).$$

Hence

$$\int_{\Delta'_{p,q}} \varphi_1\cdots\varphi_{p+q} = \sum_{\sigma\in\mathfrak{S}_{p,q}} \int_0^1 \varphi_{\sigma^{-1}(1)}\cdots\varphi_{\sigma^{-1}(p+q)}.$$

Lemma 29 follows if we define the shuffle so that

$$\sum_{\sigma\in\mathfrak{S}_{p,q}} \varphi_{\sigma^{-1}(1)}\cdots\varphi_{\sigma^{-1}(p+q)} = \varphi_1\cdots\varphi_p ш \varphi_{p+1}\cdots\varphi_{p+q}.$$

□

7.3 The shuffle ш and the shuffle Algebra $\mathfrak{H}_ш$

Let X be a set and K a field. On $K\langle X\rangle$ we define the *shuffle* product as follows. On the words, the map $ш : X^*\times X^*\to\mathfrak{H}$ is defined by the formula

$$(x_1\cdots x_p)ш(x_{p+1}\cdots x_{p+q}) = \sum_{\sigma\in\mathfrak{S}_{p,q}} x_{\sigma^{-1}(1)}\cdots x_{\sigma^{-1}(p+q)},$$

where $\mathfrak{S}_{p,q}$ denotes the set of permutation σ on $\{1,\dots,p+q\}$ satisfying

$$\sigma(1)<\sigma(2)<\cdots<\sigma(p) \quad\text{and}\quad \sigma(p+1)<\sigma(p+2)<\cdots<\sigma(p+q).$$

This set $\mathfrak{S}_{p,q}$ has $(p+q)!/p!q!$ elements; if $(p,q) \neq (0,0)$, it is the disjoint union of two subsets, the first one with $(p-1+q)!/(p-1)!q!$ elements consists of those σ for which $\sigma(1) = 1$, and the second one with $(p+q-1)!/p!(q-1)!$ elements consists of those σ for which $\sigma(p+1) = 1$.

Write $y_i = x_{\sigma^{-1}(i)}$, so that $x_j = y_{\sigma(j)}$ for $1 \le i \le p+q$ and $1 \le j \le p+q$. The letters $x_1, \dots, x_{p+q}$ and $y_1, \dots, y_{p+q}$ are the same, only the order may differ. However $x_1, \dots, x_p$ (which is the same as $y_{\sigma(1)}, \dots, y_{\sigma(p)}$) occur in this order in $y_1, \dots, y_{p+q}$, and so do $x_{p+1}, \dots, x_{p+q}$ (which is the same as $y_{\sigma(p+1)}, \dots, y_{\sigma(p+q)}$).

Accordingly, the previous definition of $ш : X^* \times X^* \to \mathfrak{H}$ is equivalent to the following inductive one:

$$e ш w = w ш e = w \quad \text{for any } w \in X^*,$$

and

$$(xu) ш (yv) = x\bigl(u ш (yv)\bigr) + y\bigl((xu) ш v\bigr)$$

for x and y in X (letters), u and v in X^* (words).

Example. *For k and ℓ non-negative integers and $x \in X$,*

$$x^k ш x^\ell = \frac{(k+\ell)!}{k!\ell!} x^{k+\ell}.$$

From

$$\mathfrak{S}_{2,2} = \bigl\{(1)\ ;\ (2,3)\ ;\ (2,4,3)\ ;\ (1,2,3)\ ;\ (1,2,4,3)\ ;\ (1,3)(2,4)\bigr\}$$

one deduces

$$x_1x_2 ш x_3x_4 = x_1x_2x_3x_4 + x_1x_3x_2x_4 + x_1x_3x_4x_2 + x_3x_1x_2x_4 + x_3x_1x_4x_2 + x_3x_4x_1x_2,$$

hence

$$x_0x_1 ш x_0x_1 = 2x_0x_1x_0x_1 + 4x_0^2x_1^2.$$

In the same way the relation

$$x_0x_1 ш x_0^2x_1 = x_0x_1x_0^2x_1 + 3x_0^2x_1x_0x_1 + 6x_0^3x_1^2$$

is easily checked by computing more generally $x_0x_1 ш x_2x_3x_4$ as a sum of $6!/(2!3! = 10$ *terms.*

Notice that the shuffle product of two words is most often not a word but a polynomial in $K\langle X\rangle$. We extend the definition of $ш : X^* \times X^* \to \mathfrak{H}$ to $ш : \mathfrak{H} \times \mathfrak{H} \to \mathfrak{H}$ by distributivity with respect to addition:

$$\sum_{u \in X^*} (S|u)u \ ш \sum_{v \in X^*} (T|v)v = \sum_{u \in X^*} \sum_{v \in X^*} (S|u)(T|v) u ш v.$$

One checks that the shuffle ш endows $K\langle X\rangle$ with a structure of commutative K-algebra.

From now on we consider only the special case $X = \{x_0, x_1\}$. The set $K\langle X\rangle$ with the shuffle law ш is a commutative algebra which will be denoted by $\mathfrak{H}_ш$. Since $\mathfrak{H}^1$ as well as $\mathfrak{H}^0$ are stable under ш, they define subalgebras

$$\mathfrak{H}^0_ш \subset \mathfrak{H}^1_ш \subset \mathfrak{H}_ш.$$

Since

$$\widehat{\zeta}(y_{\underline{s}})\widehat{\zeta}(y_{\underline{s}'}) = \widehat{\zeta}(y_{\underline{s}} ш y_{\underline{s}'}),$$

we deduce:

Theorem 30. *The map*

$$\widehat{\zeta} : \mathfrak{H}^0_ш \longrightarrow \mathbf{R}$$

is a homomorphism of commutative algebras.

8 Product of series and the harmonic algebra

8.1 The stuffle $\star$ and the harmonic algebra $\mathfrak{H}_\star$

There is another shuffle-like law on $\mathfrak{H}$, called the *harmonic product* by M. Hoffman and *stuffle* by other authors, denoted with a star, which also gives rise to subalgebras

$$\mathfrak{H}^0_\star \subset \mathfrak{H}^1_\star \subset \mathfrak{H}_\star.$$

The starting point is the observation that the product of two multizeta series is a linear combination of multizeta series. Indeed, the cartesian product

$$\big\{(n_1, \dots, n_k)\ ;\ n_1 > \cdots > n_k\big\} \times \big\{(n'_1, \dots, n'_{k'})\ ;\ n'_1 > \cdots > n'_{k'}\big\}$$

breaks into a disjoint union of subsets of the form

$$\big\{(n''_1, \dots, n''_{k''})\ ;\ n''_1 > \cdots > n''_{k''}\big\}$$

with each k'' satisfying $\max\{k, k'\}k'' \le k + k'$. The simplest example is $\zeta(2)^2 = 2\zeta(2, 2) + \zeta(4).$, a special case of Nielsen Reflexion Formula already seen in §1.2.

We write this as follows:

$$y_{\underline{s}} \star y_{\underline{s}'} = \sum_{\underline{s}''} y_{\underline{s}''}, \tag{31}$$

where $\underline{s}''$ runs over the tuples $(s''_1, \dots, s''_{k''})$ obtained from $\underline{s} = (s_1, \dots, s_k)$ and $\underline{s}' = (s'_1, \dots, s'_{k'})$ by inserting, in all possible ways, some 0 in the string $(s_1, \dots, s_k)$ as well as in the string $(s'_1, \dots, s'_{k'})$ (including in front and at the end), so that the new strings have the same length k'', with $\max\{k, k'\} \le k'' \le k + k'$, and by adding the two sequences term by term. Here is an example:

$$\begin{array}{cccccccc}
\underline{s} & s_1 & s_2 & 0 & s_3 & s_4 & \cdots & 0 \\
\underline{s}' & 0 & s'_1 & s'_2 & 0 & s'_3 & \cdots & s'_{k'} \\
\underline{s}'' & s_1 & s_2 + s'_1 & s'_2 & s_3 & s_4 + s'_3 & \cdots & s'_{k'}.
\end{array}$$

Notice that the weight of the last string (sum of the s''_j) is the sum of the weight of $\underline{s}$ and the weight of $\underline{s}'$.

More precisely, the law $\star$ on $\mathfrak{H}$ is defined as follows. First on X^*, the map $\star : X^* \times X^* \to \mathfrak{H}$ is defined by induction, starting with

$$x_0^n \star w = w \star x_0^n = w x_0^n$$

for any $w \in X^*$ and any $n \geq 0$ (for $n = 0$ it means $e \star w = w \star e = w$ for all $w \in X^*$), and then

$$(y_s u) \star (y_t v) = y_s\big(u \star (y_t v)\big) + y_t\big((y_s u) \star v\big) + y_{s+t}(u \star v)$$

for u and v in X^*, s and t positive integers.

We shall not use so many parentheses later: in a formula where there are both concatenation products and either shuffle of star products, we agree that concatenation is always performed first, unless parentheses impose another priority:

$$y_s u \star y_t v = y_s(u \star y_t v) + y_t(y_s u \star v) + y_{s+t}(u \star v)$$

Again this law is extended to all of $\mathfrak{H}$ by distributivity with respect to addition:

$$\sum_{u \in X^*} (S|u)u \ \ \star \sum_{v \in X^*} (T|v)v = \sum_{u \in X^*} \sum_{v \in X^*} (S|u)(T|v) u \star v.$$

Remark. *From the definition (by induction on the length of uv) one deduces*

$$(u x_0^m) \star (v x_0^n) = (u \star v) x_0^{m+n}$$

for $m \geq 0$, u and v in X^*.

Example. .

$$y_s^{\star 3} = y_s \star y_s \star y_s = 6y_s^3 + 3y_s y_{2s} + 3y_{2s} y_2 + y_{3s}.$$

The set $K\langle X\rangle$ with the harmonic law $\star$ is a commutative algebra which will be denoted by $\mathfrak{H}_\star$. Since $\mathfrak{H}^1$ as well as $\mathfrak{H}^0$ are stable under $\star$, they define subalgebras

$$\mathfrak{H}_\star^0 \subset \mathfrak{H}_\star^1 \subset \mathfrak{H}_\star.$$

Since

$$\widehat{\zeta}(y_{\underline{s}})\widehat{\zeta}(y_{\underline{s}'}) = \widehat{\zeta}(y_{\underline{s}} \star y_{\underline{s}'}),$$

we deduce:

Theorem 32. *The map*

$$\widehat{\zeta} : \mathfrak{H}_\star^0 \longrightarrow \mathbf{R}$$

is a homomorphism of commutative algebras.

8.2 Regularized double shuffle relations

As a consequence of theorems 30 and 32, the kernel of $\widehat{\zeta}$ contains all elements $w ш w' - w \star w'$ for w and w' in $\mathfrak{H}^0$: indeed

$$\widehat{\zeta}(w ш w') = \widehat{\zeta}(w)\widehat{\zeta}(w') = \widehat{\zeta}(w \star w'), \quad \text{hence} \quad \widehat{\zeta}(w ш w' - w \star w') = 0.$$

However the relation $\zeta(2,1) = \zeta(3)$ (due to Euler) is not a consequence of these relations, but one may derive it in a formal way as follows.

Consider

$$y_1 ш y_2 = x_1 ш x_0 x_1 = 2x_0^2 x_1 + x_1 x_0 x_1 = 2y_2 y_1 + y_1 y_2$$

and

$$y_1 \star y_2 = y_1 y_2 + y_2 y_1 + y_3.$$

They are not in $\mathfrak{H}^0$, but their difference

$$y_1 ш y_2 - y_1 \star y_2 = y_2 y_1 - y_3$$

is in $\mathfrak{H}^0$, and Euler's relation says that this difference is in the kernel of $\widehat{\zeta}$. This is the simplest example of the so–called *Regularized double shuffle relations*.