Update: March 12, 2021

April 12 - 23, 2021: Hanoi (Vietnam) (online)
CIMPA School on Functional Equations: Theory, Practice and Interaction.

Introduction to Transcendental Number Theory **8**

Conjectures. Algebraic independence of transcendental numbers

Michel Waldschmidt

Sorbonne Université IMJ-PRG
Institut de Mathématiques de Jussieu – Paris Rive Gauche
http://www.imj-prg.fr/~michel.waldschmidt

Abstract

Schanuel's conjecture asserts that for linearly independent complex numbers $x_1, ..., x_n$, there are at least n algebraically independent numbers among the $2n$ numbers

$$x_1, \ldots, x_n, \ \exp(x_1), \ldots, \exp(x_n).$$

This simple statement has many remarkable consequences ; we will explain some of them. We will also present the state of the art on this topic.

Note : We write $\exp z$ for e^z.

Linear independence over $\mathbb{Q}$

Given complex numbers, we may ask whether they are linearly independent over $\mathbb{Q}$.

For instance given a number x, the linear independence of $1, x$ over $\mathbb{Q}$ is equivalent to the irrationality of x.

As an example, the numbers

$$\log 2, \quad \log 3, \quad \log 5, \dots \quad \log p, \dots$$

are linearly independent over $\mathbb{Q}$: for $b_i \in \mathbb{Z}$,

$$b_1 \log p_1 + \cdots + b_n \log p_n = 0 \quad \Longrightarrow b_1 = \cdots = b_n = 0.$$

$$p_1^{b_1} \cdots p_n^{b_n} = 1 \quad \Longrightarrow b_1 = \cdots = b_n = 0.$$

Linear independence over $\overline{\mathbb{Q}}$

The set of algebraic numbers is a subfield of $\mathbb{C}$ (sums and products of algebraic numbers are algebraic).

Given complex numbers, we may ask whether they are linearly independent over the field $\overline{\mathbb{Q}}$ of algebraic numbers.

For instance, given a number x, the linear independence of $1, x$ over $\overline{\mathbb{Q}}$ is equivalent to the transcendence of x.

It has been proved by A. Baker in 1968 that the numbers

$$1, \log 2, \log 3, \log 5, \dots \log p, \dots$$

are linearly independent over $\overline{\mathbb{Q}}$: for $\beta_i \in \overline{\mathbb{Q}}$,

$$\beta_0 + \beta_1 \log p_1 + \cdots + \beta_n \log p_n = 0 \implies \beta_0 = \cdots = \beta_n = 0.$$

Algebraic independence

Given complex numbers $x_1, \ldots, x_n$, we may ask whether they are algebraically independent over $\mathbb{Q}$: this means that there is no nonzero polynomial $P \in \mathbb{Q}[X_1, \ldots, X_n]$ such that $P(x_1, \ldots, x_n) = 0$.

This is equivalent to saying that $x_1, \ldots, x_n$ are algebraically independent over $\overline{\mathbb{Q}}$: if a nonzero polynomial $Q \in \overline{\mathbb{Q}}[X_1, \ldots, X_n]$ satisfies $Q(x_1, \ldots, x_n) = 0$, then by taking for P the product of the "conjugates" of Q over $\mathbb{Q}$ one gets a nonzero polynomial $P \in \mathbb{Q}[X_1, \ldots, X_n]$ such that $P(x_1, \ldots, x_n) = 0$.

For $n = 1$, x_1 is algebraically independent over $\mathbb{Q}$ if and only if x_1 is transcendental over $\mathbb{Q}$.
If $x_1, \ldots, x_n$ are algebraically independent, each of these numbers x_i is transcendental.

Transcendence degree

The transcendence degree $\operatorname{tr}\deg(K_2/K_1)$ of a field extension $K_1 \subset K_2$ is the maximal number of elements in K_2 which are algebraically independent over K_1. The transcendence degree $\operatorname{tr}\deg K$ of a field K of characteristic zero is the transcendence degree of K over $\mathbb{Q}$.

Given complex numbers $t_1, \ldots, t_m$, the maximal number of algebraically independent elements in the set $\{t_1, \ldots, t_m\}$ is the same as the transcendence degree of the field $\mathbb{Q}(t_1, \ldots, t_m)$ (over $\mathbb{Q}$).

The transcendence degree of the field $\mathbb{Q}(t_1, \ldots, t_m)$ is m if and only if $t_1, \ldots, t_m$ are algebraically independent.

For $m = 1$, the transcendence degree of the field $\mathbb{Q}(x)$ is 0 if x is algebraic, 1 if x is transcendental.

Addtivity of the transcendence degree

For $K_1 \subset K_2 \subset K_3$, we have

$$\operatorname{tr}\deg(K_3/K_1) = \operatorname{tr}\deg(K_3/K_2) + \operatorname{tr}\deg(K_2/K_1).$$

Also K_2 is an algebraic extension of K_1 if and only if $\operatorname{tr}\deg(K_2/K_1) = 0$.

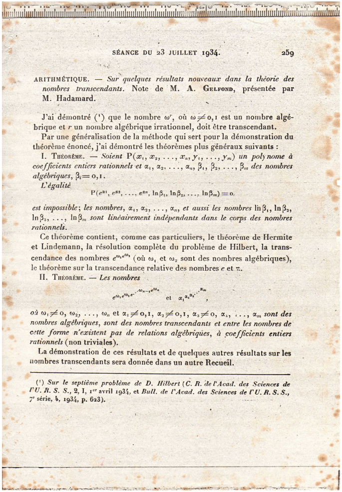
Lindemann–Weierstraß Theorem (1885)

Let $\beta_1, \ldots, \beta_n$ be algebraic numbers which are linearly independent over $\mathbb{Q}$. Then the numbers $\mathrm{e}^{\beta_1}, \ldots, \mathrm{e}^{\beta_n}$ are algebraically independent over $\mathbb{Q}$.

Ferdinand von Lindemann
(1852 – 1939)

Karl Weierstrass
(1815 - 1897)

A.O. Gel'fond CRAS 1934



SÉANCE DU 23 JUILLET 1934. 259

ARITHMÉTIQUE. — *Sur quelques résultats nouveaux dans la théorie des nombres transcendants.* Note de M. A. **Gelfond**, présentée par M. Hadamard.

J'ai démontré (¹) que le nombre ω^r, où $\omega \neq 0, 1$ est un nombre algébrique et r un nombre algébrique irrationnel, doit être transcendant.

Par une généralisation de la méthode qui sert pour la démonstration du théorème énoncé, j'ai démontré les théorèmes plus généraux suivants :

I. Théorème. — *Soient* $P(x_1, x_2, \ldots, x_n, y_1, \ldots, y_m)$ *un polynome à coefficients entiers rationnels et* $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_m$ *des nombres algébriques,* $\beta_i = 0, 1$.

L'égalité

$$P(e^{\alpha_1}, e^{\alpha_2}, \ldots, e^{\alpha_n}, \ln\beta_1, \ln\beta_2, \ldots, \ln\beta_m) = 0$$

est impossible; les nombres, $\alpha_1, \alpha_2, \ldots, \alpha_n$, *et aussi les nombres* $\ln\beta_1, \ln\beta_2, \ln\beta_3, \ldots, \ln\beta_m$ *sont linéairement indépendants dans le corps des nombres rationnels.*

Ce théorème contient, comme cas particuliers, le théorème de Hermite et Lindemann, la résolution complète du problème de Hilbert, la transcendance des nombres $e^{\omega_1 e^{\omega_2}}$ (où ω_1 et ω_2 sont des nombres algébriques), le théorème sur la transcendance relative des nombres e et π.

II. Théorème. — *Les nombres*

$$e^{\omega_1 e^{\omega_2 e^{\cdot^{\cdot^{\cdot \omega_{n-1} e^{\omega_n}}}}}} \quad \text{et} \quad \alpha_1^{\alpha_2^{\alpha_3^{\cdot^{\cdot^{\cdot \alpha_m}}}}},$$

où $\omega_1 \neq 0, \omega_2, \ldots, \omega_n$ et $\alpha_1 \neq 0, 1$, $\alpha_2 \neq 0, 1$, $\alpha_3 \neq 0$, $\alpha_4, \ldots, \alpha_m$ *sont des nombres algébriques, sont des nombres transcendants et entre les nombres de cette forme n'existent pas de relations algébriques, à coefficients entiers rationnels* (non triviales).

La démonstration de ces résultats et de quelques autres résultats sur les nombres transcendants sera donnée dans un autre Recueil.

(¹) *Sur le septième problème de D. Hilbert* (*C. R. de l'Acad. des Sciences de l'U. R. S. S.*, 2, I, 1er avril 1934, et *Bull. de l'Acad. des Sciences de l'U. R. S. S.*, 7e série, 4, 1934, p. 623).

Statement by Gel'fond (1934)

Let $P(x_1, x_2, \ldots, x_n, y_1, \ldots, y_m)$ *be a polynomial with rational integer coefficients and* $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_m$ *algebraic numbers,* $\beta_i \neq 0, 1$*.*
The equality

$$P(\mathrm{e}^{\alpha_1}, \mathrm{e}^{\alpha_2}, \ldots, \mathrm{e}^{\alpha_n}, \ln\beta_1, \ln\beta_2, \ldots, \ln\beta_m) = 0$$

is impossible ; the numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$*, as well as the numbers* $\ln\beta_1, \ln\beta_2, \ldots, \ln\beta_m$ *are linearly independent in the rational numbers field.*

Statement by Gel'fond (1934)

This theorem includes as special cases, the theorems of Hermite and Lindemann, the complete solution of Hilbert's problem, the transcendence of numbers $e^{\omega_1 e^{\omega_2}}$ (where ω_1 and ω_2 are algebraic numbers), the theorem on the relative transcendence of the numbers e and π.

Second statement by A.O. Gel'fond

The numbers

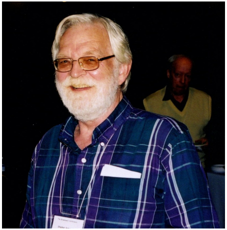
$$\mathrm{e}^{\omega_1 \mathrm{e}^{\omega_2 \mathrm{e}^{\cdots^{\omega_{n-1}\mathrm{e}^{\omega_n}}}}} \quad \textit{and} \quad \alpha_1^{\alpha_2^{\alpha_3^{\cdots^{\alpha_m}}}},$$

where $\omega_1 \neq 0, \omega_2, \ldots, \omega_n$ *and* $\alpha_1 \neq 0, 1$, $\alpha_2 \neq 0, 1$, $\alpha_3 \neq 0$, $\ldots, \alpha_m$ *are algebraic numbers, are transcendental numbers, and among numbers of this form there is no nontrivial algebraic relations with rational integer coefficients.*

The proof of this result and a few other results on transcendental numbers will be given in another journal.

Remark by Mathilde Herblot : the condition on α_2 should be that it is irrational.

Schanuel's Conjecture



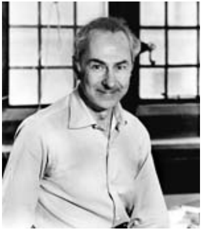
If $x_1, \ldots, x_n$ *are* $\mathbb{Q}$*–linearly independent complex numbers, then* n *at least of the* $2n$ *numbers* $x_1, \ldots, x_n$, $\mathrm{e}^{x_1}, \ldots, \mathrm{e}^{x_n}$ *are algebraically independent.*

Equivalently :
If $x_1, \ldots, x_n$*are* $\mathbb{Q}$*-linearly independent complex numbers, then*

$$\mathrm{tr}\deg\mathbb{Q}\bigl(x_1, \ldots, x_n,\ \mathrm{e}^{x_1}, \ldots, \mathrm{e}^{x_n}\bigr) \geq n.$$

Origin of Schanuel's Conjecture

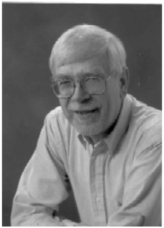
Course given by Serge Lang (1927–2005) at Columbia in the 60's



S. Lang – *Introduction to transcendental numbers*, Addison-Wesley 1966.

Formal analogs

W.D. Brownawell
(was a student of Schanuel)

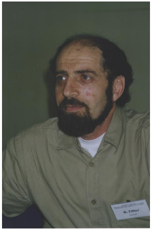


J. Ax's Theorem (1968) :
Version of Schanuel's Conjecture for power series over $\mathbb{C}$
(and R. Coleman for power series over $\overline{\mathbb{Q}}$)
Work by W.D. Brownawell and K. Kubota on the elliptic analog of Ax's Theorem.

Dale Brownawell and Stephen Schanuel

Methods from logic

Ehud Hrushovski — Boris Zilber — Jonathan Kirby


"predimension" function (E. Hrushovski)

B. Zilber : "pseudoexponentiation"

Also : A. Macintyre, D.E. Marker, G. Terzo, A.J. Wilkie, D. Bertrand. . .

Daniel Bertrand

Daniel Bertrand,

Schanuel's conjecture for non-isoconstant elliptic curves over function fields.

Model theory with applications to algebra and analysis. Vol. 1, 41–62, London Math. Soc. Lecture Note Ser., **349**, Cambridge Univ. Press, Cambridge, 2008.

Lindemann–Weierstraß Theorem (1885)

According to the Lindemann–Weierstraß Theorem, Schanuel's Conjecture is true for algebraic $x_1, \ldots, x_n$: in this case the transcendence degree of the field $\mathbb{Q}\big(x_1, \ldots, x_n,\ \mathrm{e}^{x_1}, \ldots, \mathrm{e}^{x_n}\big)$ is n.

Ferdinand von Lindemann
(1852 – 1939)

Karl Weierstrass
(1815 - 1897)

Transcendence degree $\leq n$

If we select $e^{x_1}, \ldots, e^{x_s}$ to be algebraic (this means that the x_i's are logarithms of algebraic numbers) $x_{s+1}, \ldots, x_n$ also to be algebraic, then the transcendence degree of the field

$$\mathbb{Q}(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n})$$

is the same as the transcendence degree of the field

$$\mathbb{Q}(x_1, \ldots, x_s, e^{x_{s+1}}, \ldots, e^{x_n}),$$

hence is $\leq n$. The conjecture (A.O. Gel'fond) is that it is n.

Baire and Lebesgue

René Baire
1874 – 1932

Henri Léon Lebesgue
1875 – 1941

The set of tuples $(x_1, \ldots, x_n)$ in $\mathbb{C}^n$ such that the $2n$ numbers $x_1, \ldots, x_n, \ \mathrm{e}^{x_1}, \ldots, \mathrm{e}^{x_n}$ are algebraically independent

- is a G_δ set (countable intersection of dense open sets) in Baire's classification (a *generic set* for dynamical systems)
- and has full Lebesgue measure.

True for any transcendental function in place of the exponential function.

Mathematical genealogy

René Baire (1899)

|

Arnaud Denjoy (1909)

|

Charles Pisot (1938)

|

Yvette Amice (1965)

|

Jean Fresnel (1967)

|

Michel Waldschmidt (1972)

http://genealogy.math.ndsu.nodak.edu

Joint work with Senthil Kumar and Thangadurai

Given two integers m and n with $1 \leq m \leq n$, there exist uncountably many tuples $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ such that $x_1, \ldots, x_n$ and $e^{x_1}, \ldots, e^{x_n}$ are all Liouville numbers and the transcendence degree of the field

$$\mathbb{Q}(x_1, \ldots, x_n,\ e^{x_1}, \ldots, e^{x_n})$$

is $n + m$.

$m = 0?$

$1 \leq m \leq n$:

$$\operatorname{tr\,deg} \mathbb{Q}(x_1, \dots, x_n,\ e^{x_1}, \dots, e^{x_n}) = n + m.$$

We do not know whether there are Liouville numbers x such that e^x is also a Liouville number and the two numbers x and e^x are algebraically dependent.

Schanuel's Conjecture for $n = 1$

For $n = 1$, Schanuel's Conjecture is the Hermite–Lindemann Theorem :

> *If x is a non–zero complex numbers, then one at least of the two numbers x, e^x is transcendental.*

Equivalently, *if x is a non–zero algebraic number, then e^x is a transcendental number.*

Another equivalent statement is that *if α is a non–zero algebraic number and $\log \alpha$ any non–zero logarithm of α, then $\log \alpha$ is a transcendental number.*

Consequence : transcendence of numbers like

$$\mathrm{e}, \quad \pi, \quad \log 2, \quad \mathrm{e}^{\sqrt{2}}.$$

`Proof:` take for x respectively

$$1, \quad i\pi, \quad \log 2, \quad \sqrt{2}.$$

Schanuel's Conjecture for $n = 2$

For $n = 2$, Schanuel's Conjecture is not yet known :

? If x_1, x_2 are $\mathbb{Q}$–linearly independent complex numbers, then among the 4 numbers x_1, x_2, e^{x_1}, e^{x_2}, at least two are algebraically independent.

A few consequences (open problems) :
With $x_1 = 1$, $x_2 = i\pi$: algebraic independence of e and π.
With $x_1 = 1$, $x_2 = e$: algebraic independence of e and e^e.
With $x_1 = \log 2$, $x_2 = (\log 2)^2$: algebraic independence of $\log 2$ and $2^{\log 2}$.
With $x_1 = \log 2$, $x_2 = \log 3$: algebraic independence of $\log 2$ and $\log 3$.

Baker's linear independence Theorem

Let $\lambda_1, \ldots, \lambda_n$ be $\mathbb{Q}$–linearly independent logarithms of algebraic numbers. Then the numbers $1, \lambda_1, \ldots, \lambda_n$ are linearly independent over the field $\overline{\mathbb{Q}}$ of algebraic numbers.

Schanuel's Conjecture deals with algebraic independence (over $\mathbb{Q}$ or $\overline{\mathbb{Q}}$), Baker's Theorem deals with linear independence. Baker's Theorem is a special case of Schanuel's Conjecture.

Serre's reformulation of Baker's Theorem

Denote by $\mathcal{L}$ the set of complex numbers λ for which e^{λ} is algebraic (set of logarithms of algebraic numbers). Hence $\mathcal{L}$ is a $\mathbb{Q}$-vector subspace of $\mathbb{C}$.

J-P. Serre
(Bourbaki seminar) :
the injection of $\mathcal{L}$ into $\mathbb{C}$ extends to a $\overline{\mathbb{Q}}$–linear map $\iota : \overline{\mathbb{Q}} + \mathcal{L} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \to \mathbb{C}$, and Baker's Theorem means that ι is an injective map.

Algebraic independence

Towards Schanuel's Conjecture :

Ch. Hermite, F. Lindemann, C.L. Siegel, A.O. Gel'fond, Th. Schneider, A. Baker, S. Lang, W.D. Brownawell, D.W. Masser, D. Bertrand, G.V. Chudnovsky, P. Philippon, G. Wüstholz, Yu.V. Nesterenko, D. Roy. . .

Algebraic independence : A.O. Gel'fond 1948

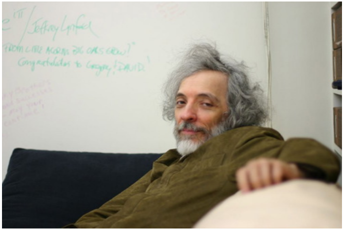
The two numbers $2^{\sqrt[3]{2}}$ and $2^{\sqrt[3]{4}}$ are algebraically independent.

More generally, if α is an algebraic number, $\alpha \neq 0$, $\alpha \neq 1$ and if β is an algebraic number of degree $d \geq 3$, then two at least of the numbers

$$\alpha^{\beta},\ \alpha^{\beta^2},\ \dots,\alpha^{\beta^{d-1}}$$

are algebraically independent.

Algebraic independence



G.V. Chudnovsky (1978)

The numbers π and $\Gamma(1/4) = 3.625\,609\,908\,2\ldots$ are algebraically independent.

Also π and $\Gamma(1/3) = 2.678\,938\,534\,7\ldots$ are algebraically independent.

On the number e^π

Yu.V.Nesterenko (1996)
Algebraic independence of $\Gamma(1/4)$, π and e^π.
Also : Algebraic independence of $\Gamma(1/3)$, π and $e^{\pi\sqrt{3}}$.

Corollary : *The numbers* $\pi = 3.141\,592\,653\,5\ldots$ *and* $e^\pi = 23.140\,692\,632\,7\ldots$ *are algebraically independent.*

The proof uses modular functions.

On the number e^π

Open problem : e^π is not a Liouville number :

$$\left| e^\pi - \frac{p}{q} \right| > \frac{1}{q^\kappa}.$$

Algebraic independence of π and e^π : Nesterenko

Chudnosvki : algebraic independence of π and $\Gamma(1/4)$

Nesterenko : Algebraic independence of π, $\Gamma(1/4)$ and e^π

Open problem : algebraic independence of π and e.

Expected : e, π and e^π are algebraic independent.

Easy consequence of Schanuel's Conjecture

A consequence of Schanuel's Conjecture is that the following numbers are algebraically independent :

$$e+\pi,\ e\pi,\ \pi^{e},\ e^{\pi^2},\ e^{e},\ e^{e^2},\dots,\ e^{e^{e}},\dots,\ \pi^{\pi},\pi^{\pi^2},\dots\ \pi^{\pi^{\pi}}\dots$$

$$\log\pi,\ \log(\log 2),\ \pi\log 2,\ (\log 2)(\log 3),\ 2^{\log 2},\ (\log 2)^{\log 3}\dots$$

Exercise : prove this statement using Schanuel's Conjecture several times.

Conjecture of algebraic independence of logarithms of algebraic numbers

The most important special case of Schanuel's Conjecture is :

Conjecture. *Let* $\lambda_1, \ldots, \lambda_n$ *be* $\mathbb{Q}$*-linearly independent complex numbers. Assume that the numbers* $e^{\lambda_1}, \ldots, e^{\lambda_n}$ *are algebraic. Then the numbers* $\lambda_1, \ldots, \lambda_n$ *are algebraically independent over* $\mathbb{Q}$*.*

Not yet known that the transcendence degree is ≥ 2.

Reformulation by D. Roy

Instead of taking logarithms of algebraic numbers and looking for the algebraic independence relations, D. Roy fixes a polynomial and looks at the points, with coordinates logarithms of algebraic numbers, on the corresponding hypersurface.

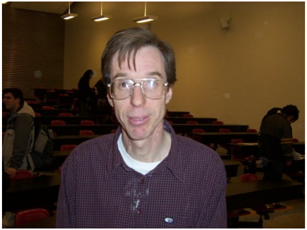
Denote by $\mathcal{L}$ the set of complex numbers λ for which e^{λ} is algebraic (logarithms of algebraic numbers). It is a $\mathbb{Q}$–vector subspace of $\mathbb{C}$.

The Conjecture on (homogeneous) algebraic independence of logarithms of algebraic numbers is equivalent to :

Conjecture (Roy). *For any algebraic subvariety V of $\mathbb{C}^n$ defined over the field $\overline{\mathbb{Q}}$ of algebraic numbers, the set $V \cap \mathcal{L}^n$ is the union of the sets $E \cap \mathcal{L}^n$, where E ranges over the set of vector subspaces of $\mathbb{C}^n$ which are contained in V.*

Points with coordinates logarithms of algebraic numbers

Damien Roy : Grassmanian varieties.



Stéphane Fischler : orbit of an affine algebraic group G over $\overline{\mathbb{Q}}$ related to a linear representation of G on a vector space with a $\overline{\mathbb{Q}}$–structure.

Quadratic relations among logarithms of algebraic numbers

One does not know yet how to prove that there is no nontrivial quadratic relations among logarithms of algebraic numbers, like

$$(\log \alpha_1)(\log \alpha_2) = \log \beta.$$

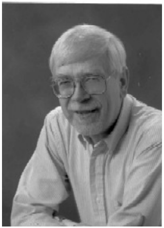
`Example:` Assume $\mathrm{e}^{\pi^2} = \beta$ is algebraic. Then

$$(-i\pi)(i\pi) = \log \beta.$$

- **Open problem :** *is the number* e^{π^2} *transcendental ?*

e^{π^2}, e and π (1972)

W.D. Brownawell
(was a student of Schanuel)



One at least of the two following statements is true :

- the number e^{π^2} is transcendental
- the two numbers e and π are algebraically independent.

Schanuel's Conjecture implies that both statements are true !

Homogeneous quadratic relations among logarithms of algebraic numbers

Any homogeneous quadratic relation among logarithms of algebraic numbers

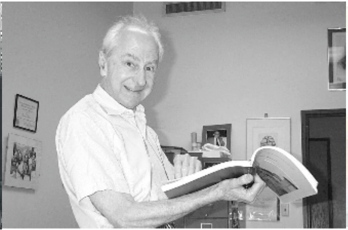
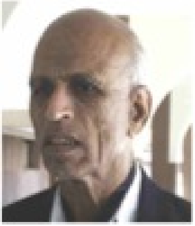
$$(\log \alpha_1)(\log \alpha_4) = (\log \alpha_2)(\log \alpha_3)$$

should be trivial.

Example of a trivial relation : $(\log 2)(\log 9) = (\log 4)(\log 3)$.

The *Four Exponentials Conjecture* can be stated as : *any quadratic relation* $(\log \alpha_1)(\log \alpha_4) = (\log \alpha_2)(\log \alpha_3)$ *among logarithms of algebraic numbers is trivial : either* $\log \alpha_1 / \log \alpha_2$ *is rational, or* $\log \alpha_1 / \log \alpha_3$ *is rational.*

S. Ramanujan, C.L. Siegel, S. Lang, K. Ramachandra

Ramanujan : Highly composite numbers.

Alaoglu and Erdős (1944), Siegel.

Four exponentials conjecture (special case)

Let t be a positive real number. Assume 2^t and 3^t are both integers. Prove that t is an integer.

Set $n = 2^t$. Then $t = (\log n)/(\log 2)$ and

$$3^t = \mathrm{e}^{t \log 3} = \mathrm{e}^{(\log n)(\log 3)/(\log 2)} = n^{(\log 3)/(\log 2)}.$$

Equivalently :

If n is a positive integer such that

$$n^{(\log 3)/(\log 2)}$$

is an integer, then n is a power of 2 :

$$2^{k(\log 3)/(\log 2)} = 3^k.$$

Damien Roy

Strategy suggested by D. Roy in 1999, Journées Arithmétiques, Roma : Conjecture equivalent to Schanuel's Conjecture.

Roy's approach to Schanuel's Conjecture (1999)

Let $\mathcal{D}$ denote the derivation

$$\mathcal{D} = \frac{\partial}{\partial X_0} + X_1 \frac{\partial}{\partial X_1}$$

over the ring $\mathbb{C}[X_0, X_1]$. The *height* of a polynomial $P \in \mathbb{C}[X_0, X_1]$ is defined as the maximum of the absolute values of its coefficients.

Let k be a positive integer, $y_1, \ldots, y_k$ complex numbers which are linearly independent over $\mathbb{Q}$, $\alpha_1, \ldots, \alpha_k$ non-zero complex numbers and s_0, s_1, t_0, t_1, u positive real numbers satisfying

$$\max\{1, t_0, 2t_1\} < \min\{s_0, 2s_1\}$$

and

$$\max\{s_0, s_1 + t_1\} < u < \frac{1}{2}(1 + t_0 + t_1).$$

Roy's Conjecture

Assume that, for any sufficiently large positive integer N*, there exists a non-zero polynomial* $P_N \in \mathbb{Z}[X_0, X_1]$ *with partial degree* $\leq N^{t_0}$ *in* X_0*, partial degree* $\leq N^{t_1}$ *in* X_1 *and height* $\leq \mathrm{e}^N$ *which satisfies*

$$\left| (\mathcal{D}^k P_N)\left(\sum_{j=1}^{k} m_j y_j, \prod_{j=1}^{k} {\alpha_j}^{m_j} \right) \right| \leq \exp(-N^u)$$

for any non-negative integers k, $m_1, \ldots, m_k$ *with* $k \leq N^{s_0}$ *and* $\max\{m_1, \ldots, m_k\} \leq N^{s_1}$*. Then*

$$\operatorname{tr}\deg \mathbb{Q}(y_1, \ldots, y_k, \alpha_1, \ldots, \alpha_k) \geq k.$$

Equivalence between Schanuel and Roy

Let $(y, \alpha) \in \mathbb{C} \times \mathbb{C}^\times$, and let s_0, s_1, t_0, t_1, u be positive real numbers satisfying the inequalities of Roy's Conjecture. *Then the following conditions are equivalent :*

(a) The number αe^{-y} is a root of unity.

(b) For any sufficiently large positive integer N, there exists a nonzero polynomial $Q_N \in \mathbb{Z}[X_0, X_1]$ with partial degree $\le N^{t_0}$ in X_0, partial degree $\le N^{t_1}$ in X_1 and height $\mathrm{H}(Q_N) \le e^N$ such that

$$\left|(\partial^k Q_N)(my, \alpha^m)\right| \le \exp(-N^u)$$

for any $k, m \in \mathbb{N}$ with $k \le N^{s_0}$ and $m \le N^{s_1}$.

Further progress by D. Roy

$\mathbb{G}_a, \quad \mathbb{G}_m, \quad \mathbb{G}_a \times \mathbb{G}_m.$

Small value estimates for the additive group. Int. J. Number Theory **6** (2010), 919–956.

Small value estimates for the multiplicative group. Acta Arith. **135** (2008), 357–393.

A small value estimate for $\mathbb{G}_a \times \mathbb{G}_m$. Mathematika **59** (2013), 333–363.

Further developments

Roy's Conjecture deals with polynomials vanishing on some subsets of $\mathbb{C} \times \mathbb{C}^{\times}$ with multiplicity along the space associated with the derivation $\partial/\partial X + Y\partial/\partial Y$.

D. Roy conjecture depends on parameters s_0, s_1, t_0, t_1, u in a certain range. D. Roy proved that if his conjecture is true for one choice of values of these parameters in the given range, then Schanuel's Conjecture is true, and that conversely, if Schanuel's Conjecture is true, then his conjecture is true for all choices of parameters in the same range.

Nguyen Ngoc Ai Van
extended the range of these parameters.

Ubiquity of Schanuel's Conjecture

Other contexts : p–adic numbers, Leopoldt's Conjecture on the p–adic rank of the units of an algebraic number field.
Non-vanishing of Regulators.
Non–degenerescence of heights.
Conjecture of B. Mazur on rational points.
Diophantine approximation on tori.

Dipendra Prasad

Gopal Prasad

Consequences of Schanuel's Conjecture

Ram Murty

Kumar Murty

N. Saradha

Purusottam Rath, Ram Murty, Sanoli Gun

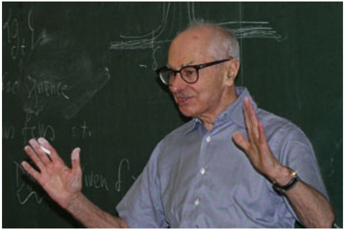
Ram and Kumar Murty (2009)

Ram Murty

Kumar Murty

Transcendental values of class group L–functions.

The Rohrlich–Lang Conjecture



The Rohrlich–Lang Conjecture implies that for any $q > 1$, the transcendence degree of the field generated by numbers

$$\pi, \quad \Gamma(a/q) \quad 1 \leq a \leq q, \ (a, q) = 1$$

is $1 + \varphi(q)/2$.

Variant of the Rohrlich–Lang Conjecture

Conjecture of S. Gun, R. Murty, P. Rath (2009) : for any $q > 1$, the numbers

$$\log \Gamma(a/q) \quad 1 \leq a \leq q, \ (a, q) = 1$$

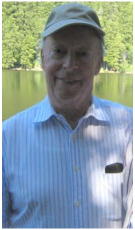
are linearly independent over the field $\overline{\mathbb{Q}}$ of algebraic numbers.

A consequence is that for any $q > 1$, there is at most one primitive odd character χ modulo q for which

$$L'(1, \chi) = 0.$$

Peter Bundschuh (1979)

.



For $p/q \in \mathbb{Q}$ with $0 < |p/q| < 1$, the sum of the series

$$\sum_{n=2}^{\infty} \zeta(n)(p/q)^n$$

is a transcendental number.

For $p/q \in \mathbb{Q} \setminus \mathbb{Z}$,

$$\frac{\Gamma'}{\Gamma}\left(\frac{p}{q}\right) + \gamma$$

is transcendental.

Peter Bundschuh (1979)

(P. Bundschuh) : *As a consequence of Nesterenko's Theorem, the number*

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1} = \frac{1}{2} + \frac{\pi}{2} \cdot \frac{e^{\pi}+e^{-\pi}}{e^{\pi}-e^{-\pi}} = 2.076\,674\,047\,4\ldots$$

is transcendental, while

$$\sum_{n=0}^{\infty} \frac{1}{n^2-1} = \frac{3}{4}$$

(telescoping series).
Hence the number

$$\sum_{n=2}^{\infty} \frac{1}{n^s-1}$$

is transcendental over $\mathbb{Q}$ for $s=4$. The transcendence of this number for even integers $s \geq 4$ would follow as a consequence of Schanuel's Conjecture.

$\sum_{n\geq 1} A(n)/B(n)$

Arithmetic nature of

$$\sum_{n\geq 1} \frac{A(n)}{B(n)}$$

where

$$A/B \in \mathbb{Q}(X).$$

In case B has distinct zeroes, by decomposing A/B in simple fractions one gets linear combinations of logarithms of algebraic numbers (Baker's method).
The example $A(X)/B(X) = 1/X^3$ shows that the general case is hard :

$$\zeta(3) = \sum_{n\geq 1} \frac{1}{n^3}.$$

S.D. Adhikari, N. Saradha, T.N. Shorey and R. Tijdeman (2001),

Open problems

Nothing is known on the arithmetic nature of *Catalan's constant*

$$G = \sum_{n\geq 1} \frac{(-1)^n}{(2n+1)^2} = 0.915\,965\,594\,177\,219\,015\,054\,603\,5\ldots$$

nor of the value

$$\Gamma(1/5) = 4.590\,843\,711\,998\,803\,053\,204\,758\,275\,929\,152\,0\ldots$$

of Euler's Gamma function, nor of the value

$$\zeta(5) = 1.036\,927\,755\,143\,369\,926\,331\,365\,486\,457\,034\,1\ldots$$

of Riemann's zeta function.

Catalan's Constant

Catalan's constant is

$$G = \sum_{n\geq 1} \frac{(-1)^n}{(2n+1)^2}$$
$$= 0.915\,965\,594\,177\,219\,015\,0\ldots$$

Is it an irrational number ?

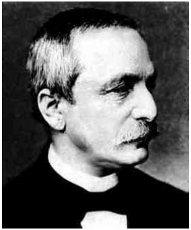
Eugène Catalan
(1814 - 1894)

Catalan's constant, Dirichlet and Kronecker

Catalan's constant is the value at $s = 2$ of the Dirichlet L–function $L(s, \chi_{-4})$ associated with the Kronecker character

$$\chi_{-4}(n) = \left(\frac{n}{4}\right) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \equiv 1 \pmod{4}, \\ -1 & \text{if } n \equiv -1 \pmod{4}. \end{cases}$$

Johann Peter Gustav Lejeune Dirichlet
1805 – 1859



Leopold Kronecker
1823 – 1891

Catalan's constant, Dedekind and Riemann

The Dirichlet L–function $L(s,\chi_{-4})$ associated with the Kronecker character χ_{-4} is the quotient of the Dedekind zeta function of $\mathbb{Q}(i)$ and the Riemann zeta function :

$$\zeta_{\mathbb{Q}(i)}(s) = L(s,\chi_{-4})\zeta(s) \qquad G = L(2,\chi_{-4}).$$

Julius Wilhelm Richard Dedekind
1831 – 1916

Georg Friedrich Bernhard Riemann
1826 – 1866

S. Gun, R. Murty, P. Rath

Assuming Schanuel's Conjecture, one at least of the two next statements is true :
(i) *The two numbers* π *and* G *are algebraically independent.*
(ii) *The number* $\Gamma_2(1/4)/\Gamma_2(3/4)$ *is transcendental.*
The multiple Gamma function of Barnes is defined by $\Gamma_0(z) = 1/z$, $\Gamma_1(z) = \Gamma(z)$,

$$\Gamma_{n+1}(z+1) = \frac{\Gamma_{n+1}(z)}{\Gamma_n(z)},$$

with $\Gamma_n(1) = 1$.

Sum of values of a rational function

Work by S.D. Adhikari, N. Saradha, T.N. Shorey and R. Tijdeman (2001),

Let P and Q be non-zero polynomials having rational coefficients and $\deg Q \geq 2 + \deg P$. Consider

$$\sum_{n \geq 0 Q(n) \neq 0} \frac{P(n)}{Q(n)}.$$

Robert Tijdeman

Sukumar Das Adhikari

N. Saradha

Telescoping series

Examples

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1, \quad \sum_{n=0}^{\infty} \frac{1}{n^2-1} = \frac{3}{4},$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{4n+1} - \frac{3}{4n+2} + \frac{1}{4n+3} + \frac{1}{4n+4} \right) = 0$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{5n+2} - \frac{3}{5n+7} + \frac{1}{5n-3} \right) = \frac{5}{6}$$

Transcendental values

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+2)} = \log 2,$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)(2n+1)(4n+1)} = \frac{\pi}{3}$$

are transcendental.

Transcendental values

$$\sum_{n=0}^{\infty} \frac{1}{(6n+1)(6n+2)(6n+3)(6n+4)(6n+5)(6n+6)}$$

$$= \frac{1}{4320}(192 \log 2 - 81 \log 3 - 7\pi\sqrt{3})$$

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1} = \frac{1}{2} + \frac{\pi}{2} \cdot \frac{e^{\pi} + e^{-\pi}}{e^{\pi} - e^{-\pi}} = 2.076\,674\,047\,4\ldots$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1} = \frac{2\pi}{e^{\pi} - e^{-\pi}} = 0.272\,029\,054\,982\ldots$$

Leonardo Pisano (Fibonacci)

The Fibonacci sequence $(F_n)_{n\geq 0}$:

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,$$

$$34,\ 55,\ 89,\ 144,\ 233\ldots$$

is defined by

$$F_0 = 0,\ F_1 = 1,$$

$$F_n = F_{n-1} + F_{n-2} \quad (n \geq 2).$$

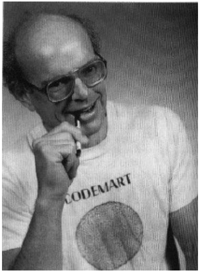
Leonardo Pisano (Fibonacci)
(1170–1250)

Encyclopedia of integer sequences (again)

$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418, 317811, 514229, 832040, 1346269, 2178309, 3524578, 5702887, 9227465, \ldots$

The Fibonacci sequence is available online
The On-Line Encyclopedia of Integer Sequences

Neil J. A. Sloane



http://oeis.org/A000045

Series involving Fibonacci numbers

The number

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2}} = 1$$

is rational, while

$$\sum_{n=0}^{\infty} \frac{1}{F_{2^n}} = \frac{7-\sqrt{5}}{2}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+1}} = \frac{1-\sqrt{5}}{2}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}+1} = \frac{\sqrt{5}}{2}$$

are irrational algebraic numbers.

Series involving Fibonacci numbers

The numbers

$$\sum_{n=1}^{\infty} \frac{1}{F_n^2}, \quad \sum_{n=1}^{\infty} \frac{1}{F_n^4}, \quad \sum_{n=1}^{\infty} \frac{1}{F_n^6},$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n^2}, \quad \sum_{n=1}^{\infty} \frac{n}{F_{2n}},$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{2^n-1} + F_{2^n+1}}, \qquad \sum_{n=1}^{\infty} \frac{1}{F_{2^n+1}}$$

are all transcendental

Series involving Fibonacci numbers

Each of the numbers

$$\sum_{n=1}^{\infty} \frac{1}{F_n}, \quad \sum_{n=1}^{\infty} \frac{1}{F_n + F_{n+2}}$$

$$\sum_{n \geq 1} \frac{1}{F_1 F_2 \cdots F_n}$$

is irrational, but it is not known whether they are algebraic or transcendental.

The first challenge here is to formulate a conjectural statement which would give a satisfactory description of the situation.

The Fibonacci zeta function

For $\Re e(s) > 0$,

$$\zeta_F(s) = \sum_{n\geq 1} \frac{1}{F_n^s}$$

$\zeta_F(2)$, $\zeta_F(4)$, $\zeta_F(6)$ are algebraically independent.

Iekata Shiokawa, Carsten Elsner and Shun Shimomura (2006)

Iekata Shiokawa

Periods : Maxime Kontsevich and Don Zagier

Periods, *Mathematics unlimited—2001 and beyond*, Springer 2001, 771–808.

A *period* is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients, over domains in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients.

The number π

Basic example of a *period* :

$$\mathrm{e}^{z+2i\pi} = \mathrm{e}^{z}$$

$$2i\pi = \int_{|z|=1} \frac{dz}{z}$$

$$\begin{aligned} \pi &= \int\int_{x^2+y^2\leq 1} dxdy &= \int_{-\infty}^{\infty} \frac{dx}{1-x^2} \\ &= \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} &= 2\int_{-1}^{1} \sqrt{1-x^2}dx. \end{aligned}$$

Further examples of periods

$$\sqrt{2} = \int_{2x^2 \leq 1} dx$$

and all algebraic numbers.

$$\log 2 = \int_{1<x<2} \frac{dx}{x}$$

and all logarithms of algebraic numbers.

$$\pi = \int_{x^2+y^2 \leq 1} dxdy,$$

A product of periods is a period (subalgebra of $\mathbb{C}$), but $1/\pi$ is expected not to be a period.

Numbers which are not periods

Problem (Kontsevich–Zagier) : *To produce an explicit example of a number which is not a period.*

Several levels :

1 *analog of Cantor* : the set of periods is countable. Hence there are real and complex numbers which are not periods ("most" of them).

Numbers which are not periods

2 *analog of Liouville*

Find a property which should be satisfied by all periods, and construct a number which does not satisfies that property.

Masahiko Yoshinaga, Periods and elementary real numbers
`arXiv:0805.0349`

Compares the periods with hierarchy of real numbers induced from computational complexities.
In particular, he proves that periods can be effectively approximated by elementary rational Cauchy sequences.

As an application, he exhibits a computable real number which is not a period.

Numbers which are not periods

3 *analog of Hermite*
Prove that given numbers are not periods

Candidates : $1/\pi$, e, Euler constant.

M. Kontsevich : exponential periods

"The last chapter, which is at a more advanced level and also more speculative than the rest of the text, is by the first author only."

Relations among periods

1. Additivity

(in the integrand and in the domain of integration)

$$\int_a^b \big(f(x) + g(x)\big)dx = \int_a^b f(x)dx + \int_a^b g(x)dx,$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

2. Change of variables :

if $y = f(x)$ is an invertible change of variables, then

$$\int_{f(a)}^{f(b)} F(y)dy = \int_a^b F\big(f(x)\big)f'(x)dx.$$

Relations among periods (continued)

$\boxed{3}$ Newton–Leibniz–Stokes Formula

$$\int_a^b f'(x)dx = f(b) - f(a).$$

Conjecture of Kontsevich and Zagier

A widely-held belief, based on a judicious combination of experience, analogy, and wishful thinking, is the following

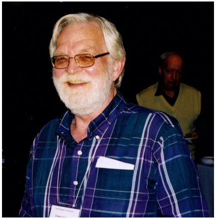
Conjecture (Kontsevich–Zagier). *If a period has two integral representations, then one can pass from one formula to another by using only rules* $\boxed{1}$, $\boxed{2}$, $\boxed{3}$ *in which all functions and domains of integration are algebraic with algebraic coefficients.*

Conjecture of Kontsevich and Zagier (continued)

In other words, we do not expect any miraculous coïncidence of two integrals of algebraic functions which will not be possible to prove using three simple rules.
This conjecture, which is similar in spirit to the Hodge conjecture, is one of the central conjectures about algebraic independence and transcendental numbers, and is related to many of the results and ideas of modern arithmetic algebraic geometry and the theory of motives.

Advice : *if you wish to prove a number is transcendental, first prove it is a period.*

Conjectures by S. Schanuel and A. Grothendieck



• Schanuel : *if* $x_1, \ldots, x_n$ *are* $\mathbb{Q}$*–linearly independent complex numbers, then* n *at least of the* $2n$ *numbers* $x_1, \ldots, x_n$*,* $e^{x_1}, \ldots, e^{x_n}$ *are algebraically independent.*

• Periods conjecture by Grothendieck : Dimension of the Mumford–Tate group of a smooth projective variety.

Motives

Y. André : generalization of Grothendieck's conjecture to motives.

Case of 1–motives : Elliptico-Toric Conjecture of C. Bertolin.

Francis Brown

Irrationality proofs for zeta values, moduli spaces and dinner parties
arXiv:1412.6508 http://www.ihes.fr/~brown/IrratModuliMotivesv8.pdf
Slides : http://www.ihes.fr/~brown/IrrationalitySlidesPrintable.pdf

A simple geometric construction on the moduli spaces $\mathcal{M}_{0,n}$ of curves of genus 0 with n ordered marked points is described which gives a common framework for many irrationality proofs for zeta values. This construction yields Apéry's approximations to $\zeta(2)$ and $\zeta(3)$, and for larger n, an infinite family of small linear forms in multiple zeta values with an interesting algebraic structure. It also contains a generalisation of the linear forms used by Ball and Rivoal to prove that infinitely many odd zeta values are irrational.

Francis Brown

For k, $s_1, \ldots, s_k$ positive integers with $s_1 \geq 2$, we set $\underline{s} = (s_1, \ldots, s_k)$ and

$$\zeta(\underline{s}) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{{n_1}^{s_1} \cdots {n_k}^{s_k}}.$$

The $\mathbb{Q}$–vector space $\mathfrak{Z}$ spanned by the numbers $\zeta(\underline{s})$ is also a $\mathbb{Q}$–algebra. For $n \geq 2$, denote by $\mathfrak{Z}_n$ the $\mathbb{Q}$-subspace of $\mathfrak{Z}$ spanned by the real numbers $\zeta(\underline{s})$ where $\underline{s}$ has weight $s_1 + \cdots + s_k = n$.

The numbers $\zeta(s_1, \ldots, s_k)$, $s_1 + \cdots + s_k = n$, where each s_i is 2 or 3, span $\mathfrak{Z}_n$ over $\mathbb{Q}$.

April 12 - 23, 2021: Hanoi (Vietnam) (online)
CIMPA School on Functional Equations: Theory, Practice and Interaction.

Introduction to Transcendental Number Theory **8**

Conjectures. Algebraic independence of transcendental numbers

Michel Waldschmidt

Sorbonne Université IMJ-PRG
Institut de Mathématiques de Jussieu – Paris Rive Gauche
http://www.imj-prg.fr/~michel.waldschmidt