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**Summer School in Analytic Number Theory
and Diophantine Approximation**

**An Introduction to
Irrationality and Transcendence Methods.**

Michel Waldschmidt

3 Auxiliary Functions in Transcendence Proofs

3.1 Explicit functions

3.1.1 Liouville

The first examples of transcendental numbers were produced by Liouville [50, 51] in 1844. At that time, it was not yet known that transcendental numbers exist. The idea of Liouville is to show that all algebraic real numbers α are badly approximated by rational numbers. The simplest example is a rational number $\alpha = a/b$: for any rational number $p/q \neq a/b$, the inequality

$$\left|\frac{a}{b} - \frac{p}{q}\right| \geq \frac{1}{bq}$$

holds. For an irrational real number x, on the contrary, for any $\epsilon > 0$ there exists a rational number p/q such that

$$\left|x - \frac{p}{q}\right| \leq \frac{1}{q^2}.$$

This yields an irrationality criterion (which is the basic tool for proving the irrationality of specific numbers), and Liouville extended it into a transcendence criterion.

The proof by Liouville involves the irreducible polynomial $f \in \mathbb{Z}[X]$ of the given irrational algebraic number α. Since α is algebraic, there exists an irreducible polynomial $f \in \mathbb{Z}[X]$ such that $f(\alpha) = 0$. Let d be the degree of f. For $p/q \in \mathbb{Q}$ the number $q^d f(p/q)$ is a non–zero rational integer, hence

$$|f(p/q)| \geq \frac{1}{q^d}.$$

On the other hand, it is easily seen that there exists a constant $c > 0$, depending only on α (and its irreducible polynomial f), such that

$$|f(p/q)| \leq c(\alpha) \left| \alpha - \frac{p}{q} \right|.$$

Therefore

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{c'(\alpha)}{q^d}$$

with $c'(\alpha) = 1/c(\alpha)$.

Let ξ be a real number such that, for any $\kappa > 0$, there exists a rational number p/q with $q \geq 2$ satisfying

$$0 < \left| \xi - \frac{p}{q} \right| < \frac{1}{q^\kappa}\cdot$$

It follows from Liouville's inequality that ξ is transcendental. Real numbers satisfying this assumption are called *Liouville's numbers.* The first examples produced by Liouville in 1844 involved continued fractions [50], immediatly after [51, 52] he considered fast convergent series.

We consider below (§ 3.3.1) extensions of Liouville's result by Thue, Siegel, Roth and Schmidt.

3.1.2 Continued fractions

Transcendental number theory is a generalization of the theory of irrational numbers. Early methods to prove that given numbers are irrational involved continued fractions. We refer to the papers by Brezinski [12, 13] for the history of this very rich theory, which can be seen as the source of Padé Approximation (§3.1.4), according to Hermite himself [38].

The proof by Lambert of the irrationality of the numbers e^r for $r \in \mathbb{Q} \setminus \{0\}$ and π in 1767 [43] rests on the continued fraction expansions of the tangent and hyperbolic tangent functions. Euler [23] studied similar continued fractions expansion, and he initiated in [24] the question of the transcendence of $2^{\sqrt{2}}$, which was to become the seventh problem of Hilbert in 1900 (see § 3.3.1).

In 1849 Hermite [37] wrote: [2]

> *Tout ce que je puis, c'est de refaire ce qu'a déjà fait Lambert, seulement d'une autre manière.*

3.1.3 Hermite

During his course at the École Polytechnique in 1815, Fourier gave a simple proof for the irrationality of e, which is now in many text books: the idea is to

[2] *All I can do is to repeat what Lambert did, just in another way.*

truncate the Taylor expansion at the origin of the exponential function. In this proof, the auxiliary function is the remainder

$$e^z - \sum_{n=0}^{N} \frac{z^n}{n!}.$$

This proof has been revisited by Liouville in 1840 [49, 48] who succeeded to extend the argument and to prove that e^2 is not a quadratic number. This result is quoted by Hermite in his memoir [38]. Fourier's argument produces rational approximations to the number e, which are sharp enough to prove the irrationality e but not the transcendence. The denominators of these approximations are $N!$. One idea of Hermite is to look for other rational approximations, and instead of the auxiliary functions $e^z - A(z)$ for $A \in \mathbb{Q}[z]$, Hermite looks at more general auxiliary functions $R(z) = B(z)e^z - A(z)$. He finds a polynomial B such that the Taylor expansion at the origin of $B(z)e^z$ has a large gap: he calls $A(z)$ the polynomial part of the expansion before the gap, so that the auxiliary function has a zero of high multiplicity at the origin. Hermite gives explicit formulae for A and B, in particular these polynomials have integer coefficients (the question is homogeneous, one may multiply by a denominator). Also he obtains upper bounds for their coefficients. As an example, given $r \in \mathbb{Q}\setminus\{0\}$ and $\epsilon > 0$ one can use this construction to show the existence of A, B and R with $0 < |R(r)| < \epsilon$. Hence $e^r \notin \mathbb{Q}$. This gives another proof of Lambert's result on the irrationality of e^r for $r \in \mathbb{Q} \setminus \{0\}$, and this proof extends to the irrationality of π as well [11, 64].

Hermite [38] goes much further, since he obtains the transcendence of e. To achieve this goal, he considers *simultaneous rational approximations to the exponential function*, in analogy with Diophantine approximation. The idea is as follows. Let $B_0, B_1, \ldots, B_m$ be polynomials in $\mathbb{Z}[x]$. For $1 \le k \le m$, define

$$R_k(x) = B_0(x)e^{kx} - B_k(x).$$

Set $b_j = B_j(1)$, $0 \le j \le m$ and

$$R = a_0 + a_1 R_1(1) + \cdots + a_m R_m(1).$$

If $0 < |R| < 1$, then $a_0 + a_1 e + \cdots + a_m e^m \neq 0$. Hermite's construction is more general: he produces rational approximations to the functions $1, e^{\alpha_1 x}, \ldots, e^{\alpha_m x}$, when $\alpha_1, \ldots, \alpha_m$ are pairwise distinct complex numbers. Let $n_0, \ldots, n_m$ be rational integers, all ≥ 0. Set $N = n_0 + \cdots + n_m$. Hermite constructs explicitly polynomials B_0, $B_1, \ldots,$ B_m with B_j of degree $N - n_j$, such that each of the functions

$$B_0(z)e^{\alpha_k z} - B_k(z), \quad (1 \le k \le m)$$

has a zero at the origin of multiplicity at least N.

Such functions are now known as *Padé approximations of the second kind* (or *of type II*).

3.1.4 Padé approximation

H.E. Padé studied systematically the approximation of complex analytic functions by rational functions.

There are two dual points of view, giving rise to the two types of *Padé Approximants* [25].

Let $f_0, \ldots, f_m$ be complex functions which are analytic near the origin and $n_0, \ldots, n_m$ be non–negative rational integers, all ≥ 0. Set $N = n_0 + \cdots + n_m$.

Padé approximants of type II are polynomials $B_0, \ldots, B_m$ with B_j having degree $\leq N - n_j$, such that each of the functions

$$B_i(z)f_j(z) - B_j(z)f_i(z) \quad (0 \leq i < j \leq m)$$

has a zero of multiplicity $> N$.

Padé approximants of the type I are polynomials $P_1, \ldots, P_m$ with P_j of degree $\leq n_j$ such that the function

$$P_1(z)f_1(z) + \cdots + P_m(z)f_m(z)$$

has a zero at the origin of multiplicity at least $N + m - 1$.

These approximants were also studied by Ch. Hermite for the exponentials functions in 1873 and 1893; later, in 1917, he gave further integral formulae for the remainder. For transcendence purposes they have been used for the first time in 1932 by K. Mahler [54, 55], who gave effective version of the transcendence theorems by Hermite, Lindemann and Weierstraß.

In the theory of Diophantine approximation, there are transference theorems, initially due to Khintchine (see for instance [15]). Similar transference properties for Padé approximation have been considered by H. Jager [40, 41] and J. Coates [17, 18, 19, 20, 21].

3.1.5 Hypergeometric methods

Explicit Padé approximations are known only for restricted classes of functions; however, when they are available, they often produce very sharp Diophantine estimates. Among the best candidates for having explicit Padé Approximations are the hypergeometric functions. Thue [79] developed this idea in the early 20th Century, and was able to solve explicitly several classes of Diophantine equations. There is a contrast between the measures of irrationality for instance which can be obtained by hypergeometric methods and those produced by other methods, like Baker's method (§ 3.3.1): typically, hypergeometric methods produce constants with one or two digits (when the expected value is something like 2), where Baker's method produces a constant with several hundreds digits. On the other hand, Baker's method works in much more general situations, and compared with the Thue–Siegel–Roth–Schmidt's method (§ 3.3.1), it has the great advantage of being explicit.

Among many contributors to this topic, we quote Thue, Baker, Chudnovskii, Bennett, Voutier, Rhin, Viola, Rivoal… These works also involve sorts of auxiliary functions (integrals) depending on parameters which needs to be suitably selected in order to produce sharp estimates.

3.2 Interpolation methods

We discuss here another type of auxiliary function which occurred in works related with a question of Weierstraß on the exceptional set of an entire function. Recall that an entire function is a complex valued function which is analytic in $\mathbb{C}$.

3.2.1 Weierstraß question

The following question was raised by Weierstraß: *Is–it true that a transcendental entire function f takes* usually *transcendental values at algebraic arguments?*

Denote by $\overline{\mathbb{Q}}$ the *field of algebraic numbers* (algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$). For an entire function f, we define the *exceptional set S_f* of f as the set of algebraic numbers α such that $f(\alpha)$ is algebraic.:

$$S_f := \bigl\{\alpha \in \overline{\mathbb{Q}}\,;\; f(\alpha) \in \overline{\mathbb{Q}}\bigr\}.$$

For instance Hermite–Lindemann's Theorem on the transcendence of $\log\alpha$ and e^β for α and β algebraic numbers is the fact that the exceptional set of the function e^z is $\{0\}$. Also the exceptional set of $e^z + e^{1+z}$ is empty, by the Theorem of Lindemann–Weierstrass. The exceptional set of functions like 2^z or $e^{i\pi z}$ is $\mathbb{Q}$, as shown by the Theorem of Gel'fond and Schneider.

The exceptional set of a polynomial is $\overline{\mathbb{Q}}$ if the polynomial has algebraic coefficients, otherwise it is finite. Also any finite set of algebraic numbers is the exceptional set of some polynomial: for $s \ge 1$ the set $\{\alpha_1, \dots, \alpha_s\}$ is the exceptional set of he polynomial $\pi(z-\alpha_1)\cdots(z-\alpha_s) \in \mathbb{C}[z]$ and also of the transcendental entire function $(z-\alpha_2)\cdots(z-\alpha_s)e^{z-\alpha_1}$.

The study of exceptional sets started in 1886 by a letter of Weierstrass to Strauss, and later developed by Strauss, Stäckel, Faber – see [57]. Further results are due to van der Poorten, Gramain, Surroca and others (see [36, 78]).

Among the results which were obtained, a typical one is the following: *if E is a countable subset of $\mathbb{C}$ and if F is a dense subset of $\mathbb{C}$, there exist transcendental entire functions f mapping E into F.*

Also there are transcendental entire functions f such that $D^k f(\alpha) \in \mathbb{Q}(\alpha)$ for all $k \ge 0$ and all algebraic α.

The question of possible sets S_f has been solved in [39]: *any set of algebraic numbers is the exceptional set of some transcendental function.* Also multiplicity can be included, as follows: define the *exceptional set with multiplicity* of a transcendental function f as the subset of $(\alpha, t) \in \overline{\mathbb{Q}} \times \mathbb{N}$ such that $f^{(t)}(\alpha) \in \overline{\mathbb{Q}}$. Here $f^{(t)}$ stands for the t-th derivative of f.

Then *any subset of $\overline{\mathbb{Q}} \times \mathbb{N}$ is the exceptional set with multiplicities of some transcendental function f.*

More generally, the main result of [39] is the following:

> *Let E be a countable subset of $\mathbb{C}$. For each pair (w, s) with $w \in E$, and $s \in \mathbb{Z}_{\ge 0}$, let $F_{w,s}$ be a dense subset of $\mathbb{C}$. Then there exists a*

transcendental entire function f such that

$$\left(\frac{d}{dz}\right)^s f(w) \in F_{w,s}$$

for all $(w, s) \in E \times \mathbb{Z}_{\geq 0}$.

One may replace $\mathbb{C}$ by $\mathbb{R}$: it means that one may take for the $F_{w,s}$ dense subsets of $\mathbb{R}$, provided that one requires E to be a countable subset of $\mathbb{R}$.

The proof is a construction of an interpolation series (see § 3.2.2) on a sequence where each w occurs infinitely often. The coefficients c_m are selected recursively in such a way that the resulting series f satisfies the required properties: the coefficients c_m are selected sufficiently small (and nonzero), so that the function f is entire and transcendental.

3.2.2 Integer valued entire functions

In 1914 Pólya initiated the study of *integer valued entire functions*; he proved that 2^z is the "smallest" entire transcendental function mapping the positive integers to rational integers. The growth of an entire function f is measured by the real valued function $R \mapsto |f|_R$, where

$$|f|_R = \sup_{|z|=R} |f(z)|.$$

More precisely, *if $f(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}_{\geq 0}$, then*

$$\limsup_{R\to\infty} 2^{-R}|f|_R \geq 1.$$

The method involves interpolation series: given an entire function f and a sequence of complex numbers $(\alpha_n)_{n\geq 0}$, define inductively a sequence $(f_n)_{n\geq 0}$ of entire functions by $f_0 = f$ and, for $n \geq 0$,

$$f_n(z) = f_n(\alpha_{n+1}) + (z - \alpha_{n+1})f_{n+1}(z).$$

Define, for $k \geq 0$,

$$P_k(z) = (z - \alpha_1)(z - \alpha_2)\cdots(z - \alpha_k).$$

One gets an expansion

$$f(z) = A(z) + P_n(z)f_n(z),$$

where

$$A = a_0 + a_1P_1 + \cdots + a_{n-1}P_{n-1} \quad \text{with} \quad a_n = f_n(\alpha_{n+1}) \quad (n \geq 0).$$

For $n \to \infty$, there are conditions for such an expansion to be convergent. See for instance [34].

There are analytic formulae for the coefficients a_n and the remainder f_n as follows. Let $x,\ z,\ \alpha_1,\dots,\alpha_n$ be complex numbers with $x \notin \{z,\ \alpha_1,\dots,\alpha_n\}$. Starting from the easy relation

$$\frac{1}{x-z} = \frac{1}{x-\alpha_1} + \frac{z-\alpha_1}{x-\alpha_1}\cdot\frac{1}{x-z}, \tag{3.1}$$

one deduces by induction the next formula due to Hermite:

$$\frac{1}{x-z} = \sum_{j=0}^{n-1} \frac{(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_j)}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_{j+1})} + \frac{(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)}\cdot\frac{1}{x-z}.$$

Let $\mathcal{D}$ be an open disc containing $\alpha_1,\dots,\alpha_n$, let $\mathcal{C}$ denote the circumference of $\mathcal{D}$, let $\mathcal{D}'$ be an open disc containing the closure of $\mathcal{D}$ and let f be an analytic function in $\mathcal{D}'$. Define

$$a_j = \frac{1}{2i\pi}\int_{\mathcal{C}} \frac{f(x)dx}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_{j+1})} \qquad (0 \le j \le n-1)$$

and

$$f_n(z) = \frac{1}{2i\pi}\int_{\mathcal{C}} \frac{f(x)dx}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)(x-z)}.$$

By means of Cauchy's residue Theorem, one deduces the so-called Newton interpolation expansion: *for any $z \in \mathcal{D}'$,*

$$f(z) = \sum_{j=0}^{n-1} a_j(z-\alpha_1)\cdots(z-\alpha_j) + (z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)f_n(z).$$

Pólya applies these formulae to prove that if f is an entire function which does not grow too fast and satisfies $f(n) \in \mathbb{Z}$ for $n \in \mathbb{Z}_{\ge 0}$, then the coefficients a_n in the expansion of f at the sequence $(\alpha_n)_{n\ge1} = \{0,1,2,\dots\}$ vanish for sufficiently large n, hence f is a polynomial.

Further works on this topic, using a variety of methods, are due to G.H. Hardy, G. Pólya, D. Sato, E.G. Straus, A. Selberg, Ch. Pisot, F. Carlson, F. Gross,... – and A.O. Gel'fond (see § 3.2.3).

3.2.3 Transcendence of e^π

Pólya's study of the growth of transcendental entire functions taking integral values at the positive rational integers was extended to the Gaussian integers by A.O. Gel'fond in 1929. By means of Newton interpolation series at the points in $\mathbb{Z}[i]$, he proved that *an entire function f which is not a polynomial and satisfies $f(\alpha) \in \mathbb{Z}[i]$ for all $\alpha \in \mathbb{Z}[i]$ satisfies*

$$\limsup_{R\to\infty} \frac{1}{R^2}\log|f|_R \ge \gamma. \tag{3.2}$$

The example of the Weierstraß sigma function (which is the Hadamard canonical product on $\mathbb{Z}[i]$), shows that the constant γ cannot be larger than $\pi/2$. In general for such problems, replacing *integer values* by *zero values* gives some hint of what should be expected, at least for the order of growth (the exponent 2 of R^2 in the left hand side of formula (3.2)), if not for the value of the constant (the number γ in the righthand side of formula (3.2)) The initial value reached by A.O. Gel'fond in 1929 was pretty small, namely $\gamma = 10^{-45}$. It was improved by several mathematicians including Fukasawa, Gruman, D.W. Masser, until 1981 when F. Gramain [35] reached $\gamma = \pi/(2e)$, which is best possible, as shown by D.W. Masser one year earlier.

This work of Gel'fond's [31] turns out to have fundamental consequences on the development of transcendental number theory, due to its connexion with the number e^π. Indeed, if the number

$$e^\pi = 23,140\,692\,632\,779\,269\,005\,729\,086\,367\,\ldots$$

is rational, then the function $e^{\pi z}$ takes values in $\mathbb{Q}(i)$ when the argument z is in $\mathbb{Z}[i]$. By expanding the function $e^{\pi z}$ into an interpolation series at the Gaussian integers, Gel'fond was able to prove the transcendence of e^π.

Further similar results where obtained just after, by means of variants of Gel'fond's argument (Kuzmin and others), but the next important step came from Siegel's introduction of further ideas in the theory (see § 3.3.1).

3.2.4 Lagrange interpolation

Newton's interpolation (§ 3.2.2) of a function yields a series of *polynomials*, namely linear combinations of products $(z-\alpha_1)\cdots(z-\alpha_n)$. Another type of interpolation has been devised in [42] by R. Lagrange (1935,) who introduced instead a series of *rational* fractions. Starting from the formula

$$\frac{1}{x-z} = \frac{\alpha-\beta}{(x-\alpha)(x-\beta)} + \frac{x-\beta}{x-\alpha}\cdot\frac{z-\alpha}{z-\beta}\cdot\frac{1}{x-z}$$

in place of (3.1), iterating and integrating as in the proof of Newton's interpolation formula, one deduces an expansion

$$f(z) = \sum_{j=0}^{n-1} b_j \frac{(z-\alpha_1)\cdots(z-\alpha_j)}{(z-\beta_1)\cdots(z-\beta_j)} + R_n(z).$$

This approach has been developed in 2006 [67] by T. Rivoal, who applies it to the Hurwitz zeta function

$$\zeta(s,z) = \sum_{k=1}^{\infty} \frac{1}{(k+z)^s} \quad (s \in \mathbb{C},\ \Re e(s) > 1,\ z \in \mathbb{C}).$$

He expands $\zeta(2,z)$ as a Lagrange series in

$$\frac{z^2(z-1)^2\cdots(z-n+1)^2}{(z+1)^2\cdots(z+n)^2}.$$

He shows that the coefficients of the expansion belong to $\mathbb{Q}+\mathbb{Q}\zeta(3)$, and is able to produce a new proof of Apéry's Theorem on the irrationality of $\zeta(3)$.

Further, he gives a new proof of the irrationality of $\log 2$ by expanding

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k+z}.$$

into a Lagrange interpolation series. Furthermore, he gives a new proof of the irrationality of $\zeta(2)$ by expanding the function

$$\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+z}\right)$$

as a Hermite–Lagrange series in

$$\frac{\bigl(z(z-1)\cdots(z-n+1)\bigr)^2}{(z+1)\cdots(z+n)}.$$

It is striking that these constructions yield exactly the same sequences of rational approximations as the one produced by methods which look very much different [27].

Further developments of the interpolation methods should be possible. For instance Taylor series are the special case of Hermite's formula with a single point and multiplicities — they give rise to Padé approximants. Multiplicities can also be introduced in R. Lagrange interpolation.

3.3 Auxiliary functions arising from the Dirichlet's box principle

3.3.1 Thue–Siegel lemma

The origin of the Thue–Siegel Lemma

The first improvement of Liouville's inequality was reached by A. Thue in 1909 [79]. Instead of evaluating the values at p/q of a polynomial in a single variable (viz. the irreducible polynomial of the given algebraic number α), he considers two approximations p_1/q_1 and p_2/q_2 of α and evaluates at the point $(p_1/q_1, p_2/q_2)$ a polynomial P in two variables. This polynomial $P \in \mathbb{Z}[X,Y]$ is constructed (or rather is shown to exist) by means of Dirichlet's box principle. The required conditions are that P has zeroes of sufficiently large order at $(0,0)$ and at $(p_1/q_1, p_2/q_2)$. The *order* is weighted (*index* of P at a point).

One of the main difficulties that Thue had to overcome was to produce a *zero estimate*, in order to find a non–zero value of some derivative.[3]

A crucial feature of Thue's argument is that he needs to select a second approximation p_2/q_2 depending on a first one p_1/q_1. Hence the method will

[3] As a matter of fact, Hermite also had some difficulty at this point of his proof [38], when he needed to check that some determinant was not 0.

not be effective unless a first very sharp approximation p_1/q_1 is required. In general one reaches a sharp estimate for all p/q with at most one exception. This approach has been worked out by J.W.S. Cassels, H. Davenport and others to deduce *upper bounds for the number of solutions of certain Diophantine equations*. However such statements are not *effective*, meaning that they do not yield complete solutions of these equations. More recently E. Bombieri has produced examples where a sufficiently sharp approximation exists for the method to work in an effective way. Later he produced *effective refinements* to Liouville's inequality by extending the argument.

Further improvement of Thue's method were obtained by C.L. Siegel in the 1920's: he developed Thue's idea and succeeded to improve his estimate. In 1929, Siegel [74], thanks to an improvement of his previous estimate, derived his well known Theorem on integer points on curves.

The introduction of the fundamental memoir [74] of C.L. Siegel in 1929 stresses the importance of Thue's idea involving the pigeonhole principle. In the second part of this fundamental paper he extends the Lindemann–Weierstraß theorem (on the algebraic independence of $e^{\beta_1}, \ldots, e^{\beta_n}$ when $\beta_1, \ldots, \beta_n$ are $\mathbb{Q}$-linearly independent algebraic numbers) from the usual exponential function to a wide class of entire functions which he calls *E-functions*. He also introduces the class of G-functions which has been extensively studied since 1929. See also his monograph in 1949 [76], Shidlovskii's book [73] for E–functions and André's book [2] for G–functions. Among interesting developments related to G functions are the works of Th. Schneider, V.G. Sprindzuck and P. Dèbes related to algebraic functions.

The work of Thue and Siegel on the rational approximation to algebraic numbers was extended by many a mathematician including Schneider, Gel'fond, Dyson, until K. F. Roth obtained in 1955 a result which is essentially optimal. For his proof he introduces polynomials with many variables.

A powerful higher dimensional generalization of Thue–Siegel–Roth's Theorem, the *Subspace Theorem*, was obtained in 1970 by W.M. Schmidt [68, 8]. Again, the proof involves a construction of an auxiliary polynomial in several variables, and one of the most difficult auxiliary result is a zero estimate (*index theorem*).

Schmidt's Subspace Theorem together with its variants (including effective estimates for the exceptional subspaces as well as results involving several valuations) have a large number of applications, to Diophantine approximation, to transcendence and also to algebraic independence – se Bilu's Bourbaki lecture [6].

Siegel, Gel'fond, Schneider

In 1932, C.L. Siegel [75] obtained the first results on the transcendence of elliptic integrals of the first kind, by means of a very ingenious argument which involved an auxiliary function whose existence follows from the Dirichlet's box principle. This idea turned out to be crucial in the development of the theory.

The seventh of the 23 problems raised by D. Hilbert in 1900 is to prove the transcendence of the numbers α^β for α and β algebraic ($\alpha \neq 0$, $\alpha \neq 1$,

$\beta \notin \mathbb{Q}$). In this statement, α^β stands for $\exp(\beta \log \alpha)$, where $\log \alpha$ is any[4] logarithm of α. The solution was achieved independently by A.O. Gel'fond [32] and Th. Schneider [69] in 1934. Consequences, already quoted by Hilbert, are the facts that $2^{\sqrt{2}}$ and e^π are transcendental.

The proofs of Gel'fond and Schneider are different, but both of them rest on some auxiliary function which arises from Dirichlet's box principle, following Siegel's contribution to the theory.

Let us argue by contradiction and assume that α, β and α^β are all algebraic, with $\alpha \neq 0$, $\alpha \neq 1$, $\beta \notin \mathbb{Q}$. Define $K = \mathbb{Q}(\alpha, \beta, \alpha^\beta)$. By assumption K is a number field.

A.O. Gel'fond's proof [32] rests on the fact that the two entire functions e^z and $e^{\beta z}$ are algebraically independent, they satisfy differential equations with algebraic coefficients and they take simultaneously values in K for infinitely many z, viz. $z \in \mathbb{Z} \log \alpha$.

Th. Schneider's proof [69] is different: he notices that the two entire functions z and $\alpha^z = e^{z \log \alpha}$ are algebraically independent, they take simultaneously values in K for infinitely many z, viz. $z \in \mathbb{Z} + \mathbb{Z}\beta$. He makes no use of differential equations, since the coefficient $\log \alpha$ which occurs by derivating the second function is not algebraic.

Schneider introduces a polynomial $A(X, Y) \in \mathbb{Z}[X, Y]$ in two variables and considers the auxiliary function

$$F(z) = A(z, \alpha^z)$$

at the points $m + n\beta$: these values γ_{mn} are in the number field $K = \mathbb{Q}(\alpha, \beta, \alpha^\beta)$.

Gel'fond also introduces also a polynomial $A(X, Y) \in \mathbb{Z}[X, Y]$ in two variables and considers the auxiliary function

$$F(z) = A(e^z, e^{\beta z});$$

the values γ_{mn} at the points $m \log \alpha$ of the derivatives $F^{(n)}(z)$ are again in the number field K.

With these notations, the proofs are similar: the first step is the existence of a non–zero polynomial A, of partial degrees bounded by L_1 and L_2, say, such that the associated numbers γ_{mn} vanish for certain values of m and n, say $0 \leq m < M$, $0 \leq n < N$. This amounts to show that a system of linear homogeneous equations has a non trivial solution, and linear algebra suffices for the existence. In this system of equations, the coefficients are algebraic numbers in the number field K, the unknowns are the coefficients of the polynomial A, and we are looking for rational integers. There are several options at this stage: one may either require only coefficients in the ring of integers of K, in which case the assumption $L_1 L_2 > MN$ suffices. An alternative is to require the coefficients to be in $\mathbb{Z}$, in which case one needs to assume $L_1 L_2 > MN[K : \mathbb{Q}]$.

[4]The assumption $\alpha \neq 1$ can be replaced by the weaker assumption $\log \alpha \neq 0$. That means that one can take $\alpha = 1$, provided that we select for $\log \alpha$ a non–zero multiple of $2i\pi$ The result allowing $\alpha = 1$ is not more general: it amounts to the same to take $\alpha = -1$ and to replace β by 2β.

This approach is not quite sufficient for the next steps: one will need estimates for the coefficients of A. This is where the Thue–Siegel's Lemma occurs into the picture: by assuming that the number of unknowns, namely $L_1 L_2$, is slightly larger than the number of equations, say twice as large, this lemma produces a bound, for a non trivial solution of the homogeneous lineara system, which is sharp enough for the rest of the proof.

The second step is an induction: one proves that γ_{mn} vanishes for further values of (m, n). Since there are two parameters (m, n), there are several options for this extrapolation, but anyway the idea is that if F has sufficiently many zeroes, then F takes rather small values on some disc (Schwarz'Lemma), its derivatives also (Cauchy's inequalities). Further, an element of K which is sufficiently small should vanish (by Liouville's inequality, or a so-called *size* inequality, or else the product formula – see [8]).

For the last step there are also several options: one may perform the induction with infinitely many steps and use an asymptotic zero estimate, or else stop after a small number of steps and prove that some determinant does not vanish. The second method is more difficult and this is the one Schneider succeeded to complete, but his proof can be simplified by pursuing the induction forever.

There is a duality between the two methods. In Gel'fond 's proof, replace L_1 and L_2 by S_1 and S_2, and replace M and N by T_0 and T_1; hence the numbers which arise are

$$\left(\frac{d}{dz}\right)^{t_0} \left(e^{(s_1+s_2\beta)z}\right)_{z=t_1 \log \alpha}$$

while in Schneider's proof replacing L_1 and L_2 by T_1 and T_2, and replacing M and N by S_0 and S_1; the numbers which arise are

$$\left(z^{t_0} \alpha^{t_1 z}\right)_{z=s_1+s_2\beta}$$

It is easily seen that they are the same, namely

$$(s_1 + s_2\beta)^{t_0} \alpha^{t_1 s_1} (\alpha^\beta)^{t_1 s_2}.$$

See [89] and § 13.7 of [92].

Gel'fond–Schneider Theorem was extended in 1966 by A. Baker [3], who proved the more general result that *if* $\log \alpha_1, \dots, \log \alpha_n$ *are* $\mathbb{Q}$*–linearly independent logarithms of algebraic numbers, then the numbers* $1, \log \alpha_1, \dots, \log \alpha_n$ *are linearly independent over* $\overline{\mathbb{Q}}$. The auxiliary function used by Baker may be considered as a function of several variables or as a function of a single complex variable, depending on the point of view (cf. [85]). The analytic estimate (Schwarz lemma) involves a single variable, but the differential equations are easier to see with several variables. See also § 3.3.1.

Indeed, assume that $\alpha_1, \dots, \alpha_n, \alpha_{n+1}, \beta_0, \dots, \beta_n$ are algebraic numbers which satisfy

$$\beta_0 + \beta_1 \log \alpha_1 + \beta_n \log \alpha_n = \log \alpha_{n+1}$$

for some speccified values of the logarithms of the α_j. Then the $n+2$ functions of n variables

$$z_0, e^{z_1}, \dots, e^{z_n}, e^{\beta_0 z_0 + \beta_1 z_1 + \cdots + \beta_n z_n}$$

satisfy differential equations with algebraic coefficients and take algebraic values at the integral multiples of the point

$$(1, \log \alpha_1, \dots, \log \alpha_n) \in \mathbb{C}^n.$$

This situation is therefore an extension of the setup in Gel'fond's solution of Hilbert's seventh problem, and Baker's method can be viewed as an extension of Gel'fond's method. The fact that all points are on a complex line $\mathbb{C}(1, \log \alpha_1, \dots, \log \alpha_n) \subset \mathbb{C}^n$ means that Baker's method requires only tools from the theory of one complex variable.

On the opposite, the corresponding extension of Schneider's method requires several variables: under the same assumptions, consider the functions

$$z_0, z_1, \dots, z_n, e^{z_0}\alpha_1^{z_1} \cdots \alpha_n^{z_n}$$

and the points in the subgroup of $\mathbb{C}^{n+1}$ generated by

$$\bigl(\{0\} \times \mathbb{Z}^n\bigr) + \mathbb{Z}(\beta_0, \beta_1, \dots, \beta_n).$$

Since Baker's Theorem includes the transcendence of e, there is no hope to prove it without introducing the differential equation of the exponential function - in order to use the fact that the last function involves e and not another number in the factor e^{z_0}, we also take derivatives with respect to z_0. For this method, we refer to [92]. We stress also the fact that this approach requires zero estimates on linear algebraic groups, which are discussed in the course by D. Roy.

Schneider–Lang Criterion

In 1949, [71] Th. Schneider produced a very general statement on algebraic values of analytic functions which can be used as *a princip for proofs of transcendence*. This statement includes a large number of previously know results like the Hermite–Lindemann and Gel'fond–Schneider Theorems. It also contains the so called *Six exponentials Theorems* [44, 92, 88, 92] (which was not explicitly in the literature then). To a certain extent, such statements provide partial answers to Weierstraß question (see § 3.2.1) that exceptional sets of a transcendental function are not too large; here one puts restrictions on the functions, while in Pólya's work concerning integer valued entire functions, the assumptions were mainly on the points and the values (the mere condition on the functions were that they have finite order of growth).

A few years latter, in his book [72] on transcendental numbers, he gave variants of this statement, which lose some generality but gain in simplicity.

Further simplifications were introduced by S. Lang in 1964 and the statement which is reproduced in his book on transcendental numbers [44] is the so-called *Criterion of Schneider–Lang* (see also the appendix of [45] as well as [85]).

Theorem 3.3. *Let K be a number field and $f_1, \dots, f_d$ be entire functions in $\mathbb{C}$. Assume that f_1 and f_2 are algebraically independent over K and of finite order of growth*[5]*. Assume also that they satisfy differential equations: for $1 \le i \le d$,*

[5] This means that the supremum of $[f(z)|$ on a disc of radius r does not grow faster than the exponential of a polynomial in r.

assume f_i' is a polynomial in $f_1, \dots, f_d$ with coefficients in K. Then the set S of $w \in \mathbb{C}$ such that all $f_i(w)$ are in K is finite[6].

This statement includes the Hermite–Lindemann Theorem on the transcendence of e^α: take

$$K = \mathbb{Q}(\alpha, e^\alpha), \quad f_1(z) = z, \quad f_2(z) = e^z, \quad S = \{m\alpha\ ;\ m \in \mathbb{Z}\},$$

as well as the Gel'fond –Schneider Theorem on the transcendence of α^β following Gel'fond's method: take

$$K = \mathbb{Q}(\alpha, \beta, \alpha^\beta), \quad f_1(z) = e^z, \quad f_2(z) = e^{\beta z}, \quad S = \{m \log \alpha\ ;\ m \in \mathbb{Z}\}.$$

This criterion does not include Schneider's method (and therefore does not include the *Six Exponentials Theorem*), but there are different criteria (not involving differential equations) for that (see for instance [44, 85, 92]).

Here is the idea of the proof of the Schneider–Lang Criterion. We argue by contradiction: assume f_1 and f_2 take simultaneously their values in the number field K for different values $w_1, \dots, w_S \in \mathbb{C}$. We want to show that there exists a non–zero polynomial $P \in \mathbb{Z}[X_1, X_2]$ such that the function $P(f_1, f_2)$ is the zero function: this will contradict the assumption that f_1 and f_2 are algebraically independent.

The first step is to show that there exists a non–zero polynomial $P \in \mathbb{Z}[X_1, X_2]$ such that $F = P(f_1, f_2)$ has a zero of high multiplicity at each w_s, $(1 \le s \le S)$: we consider the system of ST homogeneous linear equations

$$\left(\frac{d}{dz}\right)^t F(w_s) = 0 \quad \text{for} \quad 0 \le t < T,\ 1 \le s \le S,$$

where the unknowns are the coefficients of P. If we require that the partial degrees of P are bounded by L_1 and L_2, the number of unknowns is $L_1 L_2$. Since we require that P has coefficients in K we need to introduce the degree $[K : \mathbb{Q}]$ of the number field K. As soon $L_1 L_2 > TS[K : \mathbb{Q}]$, there is a non–trivial solution. Further, the Thue–Siegel Lemma produces an upper bound for the coefficients of P.

The next step is an induction: the goal is to prove that $F = 0$. One already knows

$$\left(\frac{d}{dz}\right)^t F(w_s) = 0 \quad \text{for} \quad 1 \le s \le S \quad \text{and} \quad 0 \le t < T.$$

By induction on $T' \ge T$, one proves

$$\left(\frac{d}{dz}\right)^t F(w_s) = 0 \quad \text{for} \quad 1 \le s \le S \quad \text{and} \quad 0 \le t < T'.$$

[6] For simplicity, we consider only *entire* functions; the results extends to meromorphic functions, and this is important for applications, to elliptic functions for instance. To deal with functions which are analytic in a disc only is also an interesting issue.

At the end of the induction one deduces $F = 0$, which is the desired contradiction with the algebraic independence of f_1 and f_2.

As we have seen earlier, such a sketch of proof is typical of the Gel'fond–Schneider's method.

The main analytic argument is Schwarz' Lemma for functions of one variable, which produces an upper bound for the modulus of an analytic function having many zeroes. One also require Cauchy's inequalities in order to bound the modulus of the derivatives of the auxiliary function.

In this context, a well known open problem raised by Th. Schneider (this is the second in the list of his 8 problems from his book [72]) is to prove the transcendence of $j(\tau)$ where j is the modular function defined in the upper half plane $\Im m(z) > 0$ and τ is an algebraic point in this upper half plane which is not imaginary quadratic. Schneider himself proved the transcendence of $j(\tau)$, but his proof is not direct, it rests on the use of elliptic functions (one may apply the Schneider–Lang Criterion for meromorphic functions). So the question is to prove the same result by using modular functions. In spite of recent progress on transcendence and modular functions (see § 3.3.1), this problem is still open. The difficulty lies in the analytic estimate and the absence of a suitable Schwarz' Lemma - the best results on this topic are due to I.Wakabayashi [80, 81, 82, 83, 84].

Higher dimension: several variables

In 1941, Th. Schneider [70] obtained an outstanding result on the values of Euler's Gamma and Beta function: *for any rational numbers a and b such that a, b and $a+b$ are not integers, the number $B(a,b)$ is transcendental.* His proof involves a generalization of Gel'fond's method to several variables and yields a general transcendence criterion for functions satisfying differential equations with algebraic coefficients and taking algebraic values at the points of a large Cartesian product, and he applies this criterion to the theta functions associated with the Jacobian of the Fermat curves. His transcendence results apply more generally to yield transcendence results on periods of abelian varieties.

After a suggestion of P. Cartier, S. Lang extended the classical results on the transcendence of the values of the classical exponential function to the exponential function of commutative algebraic groups. During this process, he generalized the one dimensional Schneider–Lang criterion to several variables. In this higher dimensional criterion, the conclusion is that the set of exceptional values in $\mathbb{C}^n$ cannot contain a large Cartesian product. This was the generalization to several variables of the fact that the exceptional set could not be too large.

It is interesting from an historical point of view to notice that Bertrand and Masser [5] succeeded in 1980 to deduce Baker's Theorem form the Schneider–Lang Criterion for functions of several variables with Cartesian products. They could also prove the elliptic analog of Baker's result and obtain the linear independence, over the field of algebraic numbers, of elliptic logarithms of algebraic points - at that time such a result was available only in the case of complex multiplication (by previous work of D.W. Masser). See also [92].

According to [44], Nagata suggested that the conclusion could be that this exceptional set is contained in an algebraic hypersurface, the bound for the number of points being replaced by a bound for the degree of the hypersurface. This program was completed successfully by E. Bombieri in 1970 [7]. The solution involves different tools, including L^2–estimates by L. Hörmander for functions of several variables. One main difficulty that Bombieri had to overcome was to generalize Schwarz Lemma to several variables, and his solution involves an earlier work by him and S. Lang [9], where they use Lelong's theory of the mass of zeroes of analytic functions in a ball. Chapter 7 of [88] is devoted to this question.

Similar auxiliary functions occur in the works on algebraic independence by A.O. Gel'fond [33], G.V. Chudnovskii and others. A reference is [65].

Modular functions

The solution by the team of St Etienne [4] of the problems raised by Mahler and Manin on the transcendence of the values of the modular function $J(q)$ for algebraic values of q in the unit disc[7] involves an interesting auxiliary function: the general scheme of proof is the one of Gel'fond and Schneider, but there are only two points, 0 and q. The auxiliary function F is a polynomial in z and $J(z)$, the construction by means of the Thue–Siegel Lemma requires that F vanishes at the point 0 with high multiplicity.

A similar construction was performed by Yu. V. Nesterenko in 1996 [63] when he proved the algebraic independence of π, e^π and $\Gamma(1/4)$: his main result is that *for any q in the open set $0 < |q| < 1$, the transcendence degree of the field*

$$\mathbb{Q}\big(q, P(q), Q(q), R(q)\big)$$

is at least 3. Here, P, Q, R are the classical Ramanujan functions, which are sometimes denoted as E_2, E_4 and E_6 (Eisenstein series). The auxiliary function F is a polynomial in these four functions q, $E_2(q)$, $E_4(q)$, $E_6(q)$.

3.3.2 Universal auxiliary functions

In Gel'fond–Schneider method, the auxiliary function is constructed by means of the Thue–Siegel's Lemma, and the requirement is that it has many zeroes (multiplicity are there in Gel'fond's method, not in Schneider's method). There is an alternative construction which was initiated in a joint work with M. Mignotte in 1974 [62], in connexion with quantitative statements related with transcendence criteria like the Schneider–Lang criterion. This approach turned out to be specially efficient in another context, namely in extending Schneider's method to several variables [86]. The idea is to require that the auxiliary function F has small Taylor coefficients at the origin; it follows that its modulus on some discs will be small, hence its values (included derivatives if one wishes) at points in such a disc will also be small. From Liouville's estimate one deduces that F has a lot of zeroes, more than would be reached by the Dirichlet's box principle. At this stage there are several options; the easy case is when a sharp zero estimate

[7]The connexion between $J(q)$ and $j(\tau)$ is $j(\tau) = J(e^{2i\pi\tau})$.

is known: we immediately reach the conclusion without any further extrapolation: in particular there is no need of Schwarz Lemmas in several variables. This is what happend in [86] for exponential functions in several variables, the zero estimate being due to D.W. Masser [58]. This result, dealing with products of multiplicative groups (tori), can be extended to commutative algebraic groups [87], thanks to the zero estimate of Masser and Wüstholz [59, 60].

This construction of universal auxiliary functions is developed in [89, 90]. In these paper a *dual* construction is also performed, where auxiliary *analytic functionals* are constructed, and the duality is explained by means of the Fourier-Borel transform (see § 13.7 of [92]). In the special case of exponential polynomials, it reduces to the relation

$$D^{\underline{\sigma}}_{\mathbf{w}}\left(\underline{z}^{\underline{\tau}} e^{\underline{t}\underline{z}}\right)(\underline{\eta}) = D^{\underline{\tau}}_{\mathbf{v}}\left(\underline{\zeta}^{\underline{\sigma}} e^{\underline{s}\underline{\zeta}}\right)(\underline{\xi})$$

(see [92]), which is a generalization in several variables of the formula

$$\left(\frac{d}{dz}\right)^{\sigma}\left(z^{\tau} e^{tz}\right)(\eta) = \left(\frac{d}{dz}\right)^{\tau}\left(z^{\sigma} e^{sz}\right)(\xi)$$

for t, s, ξ, η in $\mathbb{C}$ and σ, τ non–negative integers. This is how the Fourier–Borel transform provides a duality between the methods of Schneider and Gel'fond.

3.3.3 Mahler's Method

In 1929, K. Mahler [53] developed an original method to prove the transcendence of values of functions satisfying certain types of functional equations. This method was somehow forgotten, for instance it is not quoted in the 440 references of the survey paper [26] by N.I. Fel'dman, and A.B. Šidlovskiĭ. After the publication of the paper [56] by Mahler, several mathematicians extended the method (see the Lecture Notes [66] by K. Nishioka for further references). The construction of the auxiliary function is similar to what is done in Gel'fond–Schneider's method, with a main difference: in place of the Thue–Siegel Lemma, only linear algebra is required. No estimate for the coefficients of the auxiliary polynomial are needed for the rest of the proof. The numbers whose transcendence is proved by this method are not Liouville numbers, but they are quite well approximated by rational numbers.

3.4 Interpolation determinants

An interesting development of the saga of auxiliary functions took place in 1991, thanks to the introduction of interpolation determinants by M. Laurent. Its origin goes back to earlier work on a question raised by Lehmer which we first introduce.

3.4.1 Lehmer's Problem

Let θ be a non–zero algebraic integer of degree d. Mahler's *measure* of θ is

$$M(\theta) = \prod_{i=1}^{d} \max(1, |\theta_i|) = \exp\left(\int_0^1 \log|f(e^{2i\pi t}|dt\right),$$

where $\theta = \theta_1$ and $\theta_2, \cdots, \theta_d$ are the conjugates of θ and f the monic irreducible polynomial of θ in $\mathbb{Z}[X]$.

From the definition one deduces $M(\theta) \geq 1$. According to a well-known and easy result of Kronecker, $M(\theta) = 1$ *if and only if* θ *is a root of unity.*

D.H. Lehmer [47] asked whether *there is a constant* $c > 1$ *such that* $M(\theta) < c$ *implies that* θ *is a root of unity.*

Among many tools which have been introduced to answer this question, we only quote some of them which are relevant for our concern. In 1977, M. Mignotte [61] used ordinary Vandermonde determinants to study algebraic numbers whose conjugates are close to the unit circle. In 1978, C.L. Stewart [77] sharpend earlier results by Blanksby and Montgomery,... by introducing an auxiliary function (whose existence follows from the Thue–Siegel's Lemma) and using an extrapolation similar to what is done in the Gel'fond–Schneider method.

Refined estimates were obtained by E. Dobrowolski in 1979 [22] using Stewart's approach together with congruences modulo p. He achieved the best unconditional result known so far in this direction (apart from some imprrovements on the numerical value for c: *There is a constant* c *such that, for* θ *a non–zero algebraic integer of degree* d*,*

$$M(\theta) < 1 + c(\log\log d/\log d)^3$$

implies that θ *is a root of unity.*

In 1982, D. Cantor and E.G. Straus [14] revisited this method of Stewart and Dobrowolski by replacing the auxiliary function by a generalised Vandermonde determinant. The sketch of proof is the following: in Dobrowolski's proof, there is a zero lemma which can be translated into a statement that some matrix has a maximal rank; therefore some determinant is not zero. This determinant is bounded from above by means of Hadamard's inequality; the upper bound depends on $M(\theta)$. Also this determinant is shown to be big, because it has many factors of the form $\prod_{i,j} |\theta_i^p - \theta_j|^k$, for many primes p. The lower bounds makes use of a Lemma due to Dobrowolski: *For* θ *not a root of unity,*

$$\prod_{i,j} |\theta_i^p - \theta_j| \geq p^d$$

for any prime p*.*

One may also prove the lower bound by means of a p–adic Schwarz Lemma: a function (here merely a polynomial) with many zeroes has a small (p–adic) absolute value. In this case the method is similar to the earlier one, with analytic estimates on one side and arithmetic ones (Liouville type, or product formula) on the other.

Dobrowolski's result has been extended to several variables by F. Amoroso and S. David in [1] – the higher dimensional version is much more involved.

3.4.2 Laurent's interpolation determinants

In 1991, M. Laurent [46] discovered that one may get rid of the Dirichlet's box principle in Gel'fond–Schneider's method by means of his *interpolation determinants.* In the classical approach, there is a zero estimate (or vanishing estimate, also called multiplicity estimate when derivatives are there) which shows that some auxiliary function cannot have too many zeroes. This statement can be converted into the non–vanishing of some determinant. Laurent works directly with this determinant: a Liouville-type estimate produces a lower bound, the remarkable fact is that analytic estimates like Schwarz Lemma produce sharp upper bounds. Again the analytic estimate depend only in one variable (even if the determinant is a value of a function with many variables, it suffices to restrict this function to a complex line). Therefore this approach is specially efficient when dealing with functions of several variables, where Schwarz Lemma are lacking.

Interpolation determinants are easy to use when a sharp zero estimate is available. If not, it is more tricky to prove the analytic estimate. However it is possible to perform extrapolation with interpolation determinants: see [91]. Also proving algebraic independence results by means of interpolation determinants requires more work which we do not discuss here – see [65].

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3.5 Bost slope inequalities, Arakelov's Theory

Interpolation determinants require choices of bases. A further step is due to J-B. Bost in 1994, [10] where bases are no more required: the method is more intrinsic. This approach rests on Arakelov's Theory, which is used to produce *slope inequalities.* This new approach is specially interesting for results on abelian varieties obtained by transcendence methods, the example developed by Bost being the work of D. Masser and G. Wustholz on p eriods and isogenies of abelian varieties over number fields. Further estimates related to Baker's method and measures of linear independence of logarithms of algebraic points on abelian varieties have been achieved by E. Gaudron [28, 29, 30] using Bost approach. For an introduction to Bost method, we refer to the Bourbaki lecture by A. Chambert-Loir [16] in 2002.

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