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On the Maximal Length of Two Sequences of Integers in Arithmetic Progressions with the Same Prime Divisors

By

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Abstract. In this paper we consider an analogue of the problem of Erdős and Woods for arithmetic progressions. A positive answer follows from the *abc* conjecture. Partial results are obtained unconditionally.

1. For a positive integer n, we denote $u(n)$ for the greatest square free divisor of n i.e.

$$u(n) = \prod_{p|n} p,$$

$P(n)$ for the greatest prime factor of n and $\omega(n)$ for the number of distinct prime divisors of n. We understand that $u(1) = P(1) = 1$ and $\omega(1) = 0$. Erdős and Woods (see [1] for related literature) conjectured that there exists a positive integer k with the following property: if x, y are positive integers such that, for $1 \leqslant i \leqslant k$, the two numbers $x + i$ and $y + i$ have the same prime factors, then $x = y$. The solution of this problem would be of interest in logic. The only known example of positive integers (x, y, k) with $1 \leqslant x < y, k \geqslant 2$ and

$$u(x+i) = u(y+i) \quad \text{for } 1 \leqslant i \leqslant k \tag{1}$$

arise from

$$u(2^h - 2) = u(2^h(2^h - 2)), \quad u(2^h - 1) = u(2^h(2^h - 2) + 1), \quad (h \geqslant 2)$$

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and

$$u(75) = u(1215) = 15, \quad u(76) = u(1216) = 38.$$

It is proved in [1] that the relations (1) imply that

$$\log k \leqslant c_1(\log x \log\log x)^{1/2} \quad \text{for } x \geqslant 3, \tag{2}$$

$$y - x > \exp(c_2 k(\log k)^2/\log\log k) \quad \text{for } k \geqslant 3, \tag{3}$$

and

$$y - x > (k \log\log y)^{c_3 k \log\log y (\log\log\log y)^{-1}} \quad \text{for } y \geqslant 27, \tag{4}$$

where c_1, c_2, c_3 are effectively computable absolute positive constants.

In this paper, we consider an analogue of the problem of Erdös and Woods for arithmetic progressions. For each quadruple (x, y, d, d') of positive integers satisfying

$$(x, d) \neq (y, d') \quad \text{and} \quad \gcd(x, d) = \gcd(y, d') = 1, \tag{5}$$

we denote by $K = K(x, y, d, d')$ the largest positive integer K for which

$$u(x + id) = u(y + id') \quad \text{for } 0 \leqslant i \leqslant K - 1. \tag{6}$$

For instance $K(2, 2, 1, 7) = 3$ and $K(8, 4, 1, 23) = 3$, corresponding to the arithmetic progressions

$$(2, 3, 4) \text{ and } (2, 9, 16) \text{ (resp. } (8, 9, 10) \text{ and } (4, 27, 50)).$$

We observe

$$\gcd(x + id, y + id') = \gcd(x + id, d'x - dy)$$

which, together with (6) and $d'x - dy \neq 0$, implies that

$$P(x + id) = P(y + id') \leqslant |d'x - dy| \quad \text{for } 0 \leqslant i \leqslant K - 1.$$

On the other hand, we apply estimates on linear forms in logarithms to $d'(x + id) - d(y + id')$ for showing that $P(x + id) = P((x + id)(y + id'))$ tends to infinity with i, see [1, p. 228]. Thus, we secure the existence of the maximal integer K. We show that a positive answer to the arithmetic progression analogue of the problem of Erdös and Woods follows from the *abc* conjecture of Masser and Oesterlé, see LANG [3, Chap. IV §7, p. 196].

Conjecture *abc*: *For each* $\varepsilon > 0$ *three exists a constant* $\kappa(\varepsilon) > 0$ *such that if* a, b, c *are three positive rational integers with* $a + b = c$ *and*

$\gcd(a,b,c) = 1$, *then*

$$u(abc) \geqslant \kappa(\varepsilon)c^{1-\varepsilon}.$$

The following two consequences of *abc* conjecture will be proved in Section 2.

Proposition 1. *Assume the abc conjecture is true. Then for each pair* (d, d') *of positive integers, the set of pairs* (x, y) *satisfying* (5) *and* $K(x, y, d, d') > 2$ *is finite.*

Proposition 2. *Assuming the abc conjecture holds, the set of quadruples* (x, y, d, d') *satisfying* (5) *and* $K(x, y, d, d') > 4$ *is finite.*

It would be interesting to know whether there exist examples with $K(x, y, d, d') = 4$.

In Section 4, we shall prove inequalities analogous to (2), (3) and (4). We shall always assume that x, y, d and d' are positive integers satisfying (5). Further, we put

$$\Delta = \left|\frac{d'x - dy}{\gcd(d, d')}\right|.$$

By (5), we observe that $\Delta \neq 0$ and $\Delta = |x - y|$ whenever $d = d'$. We prove the next result under the assumption

$$P(x + id) = P(y + id') \quad \text{for } 0 \leqslant i < k. \tag{7}$$

This is a weaker assumption than

$$u(x + id) = u(y + id') \quad \text{for } 0 \leqslant i < k \tag{8}$$

already considered. If (7) holds, we observe $P(x + id) | \Delta$ for $0 \leqslant i < k$ which imply that $P(x + id) \leqslant \Delta$ for $0 \leqslant i < k$. If (8) holds, then $u(\prod_{i=0}^{k-1}(x + id))$ divides Δ and in particular, $u(\prod_{i=0}^{k-1}(x + id)) \leqslant \Delta$.

Proposition 3. *Assume* (7). *There exists an effectively computable absolute constant* c_4 *such that for* $k \geqslant c_4$, *we have*

$$\Delta \geqslant k^{k - \pi(k) - 3}. \tag{9}$$

We write

$$\chi_1 = x + (k-1)d, \quad \chi_2 = y + (k-1)d', \quad \chi = \max(\chi_1, \chi_2, e^e).$$

If χ is very large as compared with k, we sharpen (9) as follows.

Proposition 4(a). *If* (7) *holds with* $k \geqslant 3$, *then*

$$\log \Delta \geqslant c_5 k \log\log\log \chi, \tag{10}$$

where $c_5 > 0$ *is an effectively computable absolute constant.*

(b) *Assume* (8) *with* $k \geqslant 3$. *Then*

$$\log \Delta \geqslant c_6 k \log\log \chi \tag{11}$$

for some effectively computable absolute constant $c_6 > 0$.

From the next result, we shall deduce a lower bound for x and y in the special case $d = d'$.

Proposition 5. *Let* $k \geqslant 2$. *For* $\varepsilon > 0$, *assume that*

$$x > d^{1+\varepsilon}, \quad y > d'^{1+\varepsilon}. \tag{12}$$

If (8) *holds, then*

$$\min\left\{\log\left(\frac{d'x}{\gcd(d,d')}\right), \log\left(\frac{dy}{\gcd(d,d')}\right)\right\} \geqslant c_7(\log k)^2(\log\log k)^{-1},$$

where $c_7 > 0$ *is an effectively computable number depending only on* ε.

If $d = d'$, the assertion of Proposition 5 simplifies to

$$\min(\log x, \log y) \geqslant c_7(\log k)^2(\log\log k)^{-1}.$$

The proof of Proposition 3 is elementary and the proofs of Propositions 4 and 5 depend on the theory of linear forms in logarithms.

2. The proofs of Propositions 1 and 2 depend on the following result.

Lemma A. *Assume the abc conjecture holds. For* $0 < \varepsilon < 1$ *and for positive integers* x, d *with* $\gcd(x,d) = 1$, *we have*

$$u(x(x+d)(x+2d)) \geqslant \kappa(\varepsilon)(x+d)^{2-2\varepsilon}d^{-1} \tag{13}$$

and

$$u(x(x+d)\cdots(x+4d)) \geqslant \frac{\kappa(\varepsilon)}{8}(x+2d)^{3-3\varepsilon}. \tag{14}$$

Proof. We consider the three relatively prime positive integers

$$a = (x+d)^2 - d^2, \quad b = d^2, \quad c = (x+d)^2.$$

We observe that $a + b = c$ and

$$u(abc) = u(dx(x+d)(x+2d)) \leqslant du(x(x+d))(x+2d)).$$

Then, the inequality (13) follows immediately from *abc* conjecture.

For inequality (14), we observe that

$$a = x^2(x + 3d), \quad b = (x + d)(x + 4d)^2, \quad c = 2(x + 2d)^3$$

satisfy $a + b = c$ and write $\delta = \gcd(a, b)$. If d is even, then x is odd, $\delta = 1$ and we deduce from the *abc* conjecture that

$$2u(x(x + d)\cdots(x + 4d)) = u(2x(x + d)\cdots(x + 4d)) \geqslant \\ \geqslant \kappa(\varepsilon)2^{1-\varepsilon}(x + 2d)^{3-3\varepsilon}$$

which implies (14). If d is odd, then $\delta \mid 16$ and we apply the *abc* conjecture to

$$x^2(x + 3d)\delta^{-1}, \quad (x + d)(x + 4d)^2\delta^{-1}, \quad 2(x + 2d)^3\delta^{-1}$$

for concluding that

$$u(x(x + d)\cdots(x + 4d)) \geqslant \kappa(\varepsilon)\left(\frac{2}{\delta}\right)^{1-\varepsilon}(x + 2d)^{3-3\varepsilon} \geqslant \frac{\kappa(\varepsilon)}{8}(x + 2d)^{3-3\varepsilon}.$$

□

Proof of Proposition 1. We assume that $K > 2$. Then, we derive from (5) and (6) that $u(x(x + d)(x + 2d)) = u(y(y + d')(y + 2d'))$ divides $|d'x - dy|$. Now, we derive from (13) with $\varepsilon = 1/4$ and $d'x - dy \neq 0$ that

$$\kappa(1/4) \max\{(x + d)^{3/2}d^{-1}, (y + d')^{3/2}d'^{-1}\} \leqslant |d'x - dy|$$

which implies that $\max(x, y)$ is bounded by a number depending only on d and d'. □

Proof of Proposition 2. We assume that $K > 4$. Then, as in the proof of Proposition 1, we derive from (14) with $\varepsilon = 1/6$ that

$$\frac{\kappa(1/6)}{8} \max\{(x + 2d)^{5/2}, (y + 2d')^{5/2}\} \leqslant |d'x - dy|$$

which implies that there are only finitely many possibilities for x, y, d and d'. □

3. In this section, we shall prove lemmas for Propositions 3–5. We start with the following very useful Lemma.

Lemma 1. *For positive integers x and d with* $\gcd(x, d) = 1$, *we put* $S = \{x, x + d, \ldots, x + (k - 1)d\}$. *Let p be a prime which divides the number*

$$x(x + d)\cdots(x + (k - 1)d) = \prod_{n \in S} n.$$

Choose an $i_p \in \{0, 1, \ldots, k-1\}$ *such that*

$$|x + i_p d|_p^{-1} = \max(|n|_p^{-1}, n \in S).$$

Then

$$\prod_{i \neq i_p} |x + id|_p \geqslant |(k-1)!|_p.$$

Proof. For $i \neq i_p$ we have

$$|x + id|_p \geqslant |i - i_p|_p,$$

hence

$$\prod_{i \neq i_p} |x + id|_p \geqslant |i_p!(k-1-i_p)!|_p \geqslant |(k-1)!|_p. \qquad \square$$

Remarks. 1. This argument is due to SYLVESTER [5, Ch. 72, p. 698].

2. Since

$$|x + i_p d|_p = |\mathrm{lcm}(x, x+d, \ldots, x+(k-1)d)|_p,$$

we deduce that for positive integers x and d with $\gcd(x, d) = 1$, the number $x(x+d)\cdots(x+(k-1)d)$ divides

$$(k-1)!\,\mathrm{lcm}(x, x+d, \ldots, x+(k-1)d).$$

3. If we denote by S_1 the subset of S obtained by deleting from S all i_p with $p \leqslant k$, then

$$\prod_{n \in S_1} \prod_{p \leqslant k} |n|_p^{-1} \text{ divides } (k-1)!.$$

Compare with Erdös' result which is quoted in [1, Lemma 2.3].

Lemma 2. *For positive integers x and d with* $\gcd(x, d) = 1$, *let* S_2 *be a subset of* $\{x, x+d, \ldots, x+(k-1)d\}$. *Denote by* τ *the number of all integers* $n \in S_2$ *such that* $P(n) \leqslant k$. *Then*

$$\tau \leqslant k \frac{\log k}{\log s} + \pi(k)$$

where s is the least element of S_2.

Proof. By deleting at most $\pi(k)$ elements from the set $\{n \in S_2, P(n) \leqslant k\}$, we derive from Lemma 1 that $s^{\tau - \pi(k)} \leqslant k^k$ and the assertion follows. $\square$

The next lemma of SHOREY and TIJDEMAN [4] is on the existence of a prime divisor exceeding k of an element of $\{x, x+d, \ldots, x+ + (k-1)d\}$.

Lemma 3. *For integers* $x \geqslant 1, d \geqslant 2$ *and* $k \geqslant 3$ *with* $\gcd(x,d) = 1$, *we have*

$$P(x(x+d)\cdots(x+(k-1)d)) > k$$

unless

$$x = 2, \quad d = 7, \quad k = 3.$$

We apply Lemma 3 to derive the following estimate for Δ.

Lemma 4. *Let* $k > 6$. *For positive integers* x, y, d, d' *satisfying* $\max(d, d') > 1$, (5) *and* (7), *we have*

$$2\Delta > k^2. \tag{15}$$

Proof. We assume that $d' > 1$ and we prove (15). The proof for (15) when $d > 1$ is similar. By Lemma 3 and $k > 3$, there exists i_0 with $0 \leqslant i_0 < k$ such that $p = P(y + i_0 d') > k$. Then, we see from (7) that $p = P(x + i_0 d)$. Consequently, we derive from (5) that $p | \Delta$. If $i_0 < k/2$, we observe that $k - 1 - i_0 > (k/2) - 1 > 2$ for deriving from Lemma 3 that

$$P((y + (i_0 + 1)d')\cdots(y + (k-1)d')) \geqslant k - i_0 > k/2.$$

If $i_0 \geqslant k/2 > 3$, we apply again Lemma 3 to obtain

$$P(y(y+d')\cdots(y+(i_0-1)d')) \geqslant i_0 + 1 > k/2.$$

Consequently, there exists i_1 with $0 \leqslant i_1 < k$ and $i_1 \neq i_0$ such that $p' = P(y + i_1 d') > k/2$. As above, we derive that $p' | \Delta$. Since $p > k$ and $i_0 \neq i_1$, we observe that $p \neq p'$. Consequently $pp' | \Delta$ which implies that $\Delta \geqslant pp' > k^2/2$. □

We denote by t the number of all integers $n \in \{x, x+d, \ldots, x+ + (k-1)d\}$ such that $P(n) \leqslant k$. We observe that t is also the number of all integers $n \in \{y, y+d', \ldots, y+(k-1)d'\}$ such that $P(n) \leqslant k$ whenever (7) holds. In the next result, utilise Lemma 2 to sharpen (15) for large k.

Lemma 5. *Let* $\varepsilon > 0$. *Assume that positive integers* x, y, d, d' *satisfy* (5) *and* (7). *There exists an effectively computable number* c_8 *depending* *only on* ε *such that for* $k \geqslant c_8$, *we have*

$$\Delta \geqslant k^{(1-\varepsilon)k}. \tag{16}$$

Proof. Without loss of generality we may assume $\varepsilon < 1/3$. By (3), we may suppose that $\max(d, d') > 1$ so that (15) is valid. We may assume that k exceeds a sufficiently large effectively computable number depending only on ε. By (5), we observe that $dy \neq d'x$. We prove the lemma when $dy > d'x$ and the proof for the other case is similar. Then

$$\Delta < dy. \tag{17}$$

If $y \geqslant k^{3/2}$, we derive from Lemma 2 with $x = y$, $d = d'$ and $S_2 = \{y, y + d', \ldots, y + (k-1)d'\}$ that

$$t \leqslant \left(\frac{2}{3} + \varepsilon\right)k. \tag{18}$$

If $y < k^{3/2}$, then we see from (15) and (17) that $2d > k^{1/2}$. We apply again Lemma 2 with $S_2 = \{x + [\varepsilon k/2]d, \ldots, x + (k-1)d\}$ *and* $t \leqslant (\varepsilon k/2) + \tau$ for deriving (18). Thus, the estimate (18) is always valid. There are at least $k - t$ elements of $\{x, x + d, \ldots, x + (k-1)d\}$ whose greatest prime factors exceed k. Further, we observe from (7) that the product of these pairwise distinct at least $k - t$ primes divide Δ. Thus

$$\Delta > k^{k-t}.$$

From (18) we deduce

$$\Delta > k^{(1/3-\varepsilon)k}. \tag{19}$$

If $d > k^{k/6}$, we repeat the previous argument: we apply Lemma 2 with

$$S_2 = \{x + [\varepsilon k/2]d, \ldots, x + (k-1)d\}$$

and deduce

$$\tau \leqslant 6 + \pi(k) \quad \text{and} \quad t \leqslant \frac{\varepsilon k}{2} + \tau \leqslant \varepsilon k;$$

since $\Delta > k^{k-t}$, this implies (16).

If $d \leqslant k^{k/6}$, then from (17) and (19) we obtain

$$y > k^{k/6-\varepsilon},$$

and we use once more Lemma 2 with $x = y, d = d'$ and

$S_2 = \{y, y + d', \ldots, y + (k-1)d'\}$ to derive

$$t \leqslant \frac{6}{1-6\varepsilon} + \pi(k)$$

which again yields (16). □

Finally, we state a particular case of an estimate of GYÖRY [2] for integer solutions of Thue–Mahler equations. For distinct integers α_1, α_2 and α_3, we put

$$F(X, Y) = (X - \alpha_1 Y)(X - \alpha_2 Y)(X - \alpha_3 Y).$$

Let $H(F)$ be the maximum of the absolute values of the coefficients of F. Let $p_1, \ldots, p_s$ be distinct prime numbers with $P = \max(p_1, \ldots, p_s)$ and A some non-zero rational integer. Then

Lemma 6. *All solutions of the Thue–Mahler equation*

$$F(x, y) = A p_1^{z_1} \cdots p_s^{z_s}$$

in integers $x, y, z_1, \ldots, z_s$ *with* $\gcd(x, y) = 1$, $z_1 \geqslant 0, \ldots, z_s \geqslant 0$ *satisfy*

$$\log(\max(|x|, |y|)) \leqslant (s+1)^{C(s+1)} P^2 (1 + \log(|A| H(F))),$$

where C is an effectively computable absolute constant.

4. *Proof of Proposition 3.* We may assume that k exceeds a sufficiently large effectively computable absolute constant. We prove the result when $dy > d'x$ and the proof for the other case is similar. Then (17) is valid. For the proof of (9), we argue as in the proof of Lemma 5 for observing that it suffices to show that

$$t \leqslant \pi(k) + 3. \tag{20}$$

Let $\varepsilon = 1/25$. If $y \geqslant k^{(1/2+\varepsilon)k}$, we derive from Lemma 2 that $t \leqslant \pi(k) + 1$. Thus, we may assume that $y < k^{(1/2+\varepsilon)k}$ which, together with (16) and (17), implies that $d > k^{(1/2-2\varepsilon)k}$. Now, we apply again Lemma 2 for $S_2 = \{x + d, \ldots, x + (k-1)d\}$ to conclude (20). □

Proof of Proposition 4. We write $c_9, \ldots, c_{13}$ for effectively computable absolute constants. By applying Lemma 6 to $F(x, d)$ and $F(y, d')$ where F is the binary form $X(X + Y)(X + 2Y)$, the estimate (10) follows from (7) and the estimate (11) follows from (8) whenever k is bounded. Therefore, we may assume that k exceeds a sufficiently large effectively computable absolute constant. Further, we may suppose that

$$k < \log \chi \tag{21}$$

otherwise Proposition 4 follows from Proposition 3. We assume that $\chi_2 \geqslant \chi_1$ and the proof for the other case is similar. We write

$$y + id' = M_i M_i' \quad \text{for } 0 \leqslant i \leqslant k$$

where $P(M_i) \leqslant k$ and every prime factor of M_i' exceeds k. By Lemma 2 and (21), the set of i with $0 \leqslant i \leqslant k$ and $M_i' = 1$ has at most $\pi(k) + 2\log k$ elements. Define $I = \{i; 0 \leqslant i < k, M_i > k^{20}\}$. Using Lemma 1, we deduce that there is a subset J of I with Card $J \geqslant$ Card I $- \pi(k)$ and

$$\prod_{i \in J} \prod_{p \leqslant k} |y + id'|_p^{-1} \leqslant k!.$$

The left-hand side is nothing else than $\prod_{i \in J} M_i$, hence is at least $k^{20 \operatorname{Card} J}$. Therefore

$$\operatorname{Card} J \leqslant \frac{k}{20} \quad \text{and} \quad \operatorname{Card} I \leqslant \pi(k) + \frac{k}{20}.$$

Therefore, we find integers i_j with $1 \leqslant j \leqslant \mu$ and $\mu = [9k/10]$ in $[0, k)$ such that

$$M_{i_j} \leqslant k^{20}, M_{i_j}' > k \quad \text{for } 1 \leqslant j \leqslant \mu.$$

Furthermore, we may suppose that there are at least $[3k/4]$ integers j such that

$$P(M_{i_j}') \leqslant (\log \chi)^{1/3},$$

otherwise Proposition 4 follows from (7). We arrange these integers as $i_1 < i_2 < \cdots < i_{\mu'}$ with $\mu' = [3k/4]$. Let j be an integer in the range $1 \leqslant j \leqslant \mu' - 2$; we apply Lemma 6 to $F(y, d')$, where F is the binary form

$$F(X, Y) = (X + i_j Y)(X + i_{j+1} Y)(X + i_{j+2} Y),$$

with

$$A = M_{i_j} M_{i_{j+1}} M_{i_{j+2}}, \quad s = \omega(M_{i_j}' M_{i_{j+1}}' M_{i_{j+2}}')$$

to conclude that

$$\omega(M_{i_j}') \geqslant c_9 (\log\log \chi)(\log\log\log \chi)^{-1}$$

for at least $[k/8]$ integers j with $1 \leqslant j \leqslant \mu'$. Therefore

$$P(M'_{ij}) \geqslant c_{10} \log\log\chi \tag{22}$$

for at least $[k/8]$ integers j with $1 \leqslant j \leqslant \mu'$ and

$$\sum_{j=1}^{\mu'} \omega(M'_{ij}) \geqslant c_{11} k (\log\log\chi)(\log\log\log\chi)^{-1}. \tag{23}$$

If (7) holds, we derive from (22) that

$$(c_{10}\log\log\chi)^{c_{12}k} \leqslant \prod_{j=1}^{\mu'} P(M'_{ij}) \leqslant \Delta$$

which implies (10). If (8) holds, we obtain from (23) that

$$(\log\chi)^{c_{13}k} \leqslant u\left(\prod_{j=1}^{\mu'} M'_{ij}\right) \leqslant \Delta$$

and (11) follows. □

Proof of Proposition 5.
Put $D = \gcd(d, d')$ and $W = \sum_{i=0}^{k-1} \omega(y + id')$. We prove Proposition 5 under the assumption $d'x < dy$, i.e. we prove

$$\log\left(\frac{d'x}{D}\right) \geqslant c_7 (\log k)^2 (\log\log k)^{-1}. \tag{24}$$

The proof of Proposition 5 for the other case $d'x > dy$ is similar. We may assume that k exceeds a sufficiently large effectively computable absolute constant, otherwise (24) follows immediately. Then, we derive from Proposition 3 and $d'x < dy$ that

$$k^{k - \pi(k) - 3} \leqslant \Delta < \frac{dy}{D}.$$

We may assume that $d/D < k^{\pi(k)}$, otherwise (24) follows from (12). Thus

$$y > k^{k - 2\pi(k) - 3}. \tag{25}$$

Let $0 < \varepsilon_1 < 1$. We assume that

$$\log(x + (k-1)d) < \varepsilon_1 \log k \frac{\log\log y}{\log\log\log y} \tag{26}$$

and we shall choose ε_1 a suitable effectively computable absolute constant for arriving at a contradiction. From this contradiction, we see from (12) and (25) that

$$\log\left(\frac{d'x}{D}\right) \geqslant \log x \geqslant \varepsilon_1 (\log k)^2 (\log\log k)^{-1} - \log k,$$

implying (24).

For an integer i with $0 \leqslant i < k$, we denote by $\omega'(y + id')$ the number of prime divisors of $y + id'$ which are greater than k. From (8) we deduce $\omega'(y + id') = \omega'(x + id)$. Since

$$k^{\omega'(y+id')} \leqslant x + id \leqslant x + (k-1)d,$$

we deduce from (26) that

$$\omega'(y + id') \leqslant \varepsilon_1 \frac{\log\log y}{\log\log\log y}.$$

Also, for $0 \leqslant i < k$, we have

$$\log P(y + id') \leqslant \log k \frac{\log\log y}{\log\log\log y}.$$

Now, in view of (12), we apply the theory of linear forms in logarithms as in the proof of Proposition 4.11 of [1] for deriving

$$W \leqslant c_{14} k \log\log k + \varepsilon_1 k \frac{\log\log y}{\log\log\log y}$$

and

$$W \geqslant c_{15} k \frac{\log\log y}{\log\log\log y},$$

where c_{14} and c_{15} are positive effectively computable absolute constants. Finally, we choose $\varepsilon_1 = (2c_{15})^{-1}$ to obtain from the estimates for W that

$$c_{15} \frac{\log\log y}{\log\log\log y} < 2c_{14} \log\log k,$$

which, by (25), is not possible if k is sufficiently large. □

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References

[1] Balasubramanian, R., Shorey, T. N., Waldschmidt, M.: On the maximal length of two sequences of consecutive integers with the same prime divisors. Acta Math. Hung. **54**, 225–236 (1989).
[2] Győry, K.: Explicit upper bounds for the solutions of some Diophantine equations. Ann. Acad. Sci. Fenn. Ser. A. I. **5**, 3–12 (1980).
[3] Lang, S.: Algebra, 3rd edn. Reading, Mass.: Addison Wesley. 1993.
[4] Shorey, T. N., Tijdeman, R.: On the greatest prime factor of an arithmetical progression. In: A Tribute to Paul Erdős (A. Baker, B. Bollobas, A. Hajnal, eds.) pp. 385–389. Cambridge: Univ. Press. 1990.
[5] Sylvester, J. J.: On arithmetical series. Messenger of Math. **21**, 1–19 and 87–120; Math. Papers IV, pp. 686–731. Cambridge. 1912.

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