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Algebraic Independence of Transcendental Numbers. Gel'fond's Method and Its Developments

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The first result on algebraic independence of transcendental numbers was proved one century ago by Lindemann and Weierstrass: if $\alpha_1, \ldots, \alpha_n$ are algebraic numbers which are linearly independent over $\mathbb{Q}$, then $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are algebraically independent.

One knows four methods to derive the algebraic independence of transcendental numbers.

1 The construction, by Liouville, of transcendental numbers can be generalized to the construction of algebraically independent numbers. Several examples of algebraically free subsets of $\mathbb{C}$ with the power of continuum have been exhibited (J. von Neumann, O. Perron, H. Kneser, W. M. Schmidt, F. Kuiper and J. Popken, A. S. Fraenkel, M. G. de Bruin, A. Durand, P. Bundschuh and R. Wallisser, W. W. Adams, F. J. Wylegala, I. Shiokawa, Zhu Yao Chen, ...).

On the other hand, quantitative estimates (transcendence, linear independence or algebraic independence measures) enable one also to construct algebraically independent numbers (D. D. Mordoukhay-Boltovskoy, K. Mahler, A. O. Gel'fond, N. I. Fel'dman, S. Lang, A. A. Shmelev, A. I. Galochkin, W. D. Brownawell, K. Väänänen, M. Laurent, A. Bijlsma, F. J. Wylegala, ...).

2 In 1929, C. L. Siegel introduced a new method which enabled him to generalize the Lindemann-Weierstrass theorem to a class of entire functions (which he called E-functions) satisfying linear differential equations. Some restrictions in the hypotheses of Siegel's results were relaxed by A. B. Shidlovskii in 1953, and during the last thirty years many publications have been devoted to this subject (A. B. Shidlovskii, S. Lang, V. A. Oleinikov, T. V. Pershikova, I. I. Belogrivov, K. Mahler, A. I. Galochkin, V. G. Sprindzuck, Ju. V. Nesterenko, A. A. Shmelev, M. S. Nurmagomedov, K. Väänänen, V. H. Salihov, W. D. Brownawell, ...).[1]).

[1]) cf. A. B. Shidlovskii. *Diophantine approximations and transcendental numbers* (in russian); Izd. Mosk. Univ., МГУ 1982.

3 Also in 1929, K. Mahler studied transcendental functions satisfying certain functional equations, and succeeded to prove in certain cases the algebraic independende of their values. In the first issue of the Journal of Number Theory, in 1969, Mahler remarked that these old papers had been forgotten. Starting in 1976, new progress has been achieved with this method (K. K. Kubota, J. H. Loxton and A. J. van der Poorten, Y. Z. Flicker, D. W. Masser).

4 The fourth method, created by A. O. Gel'fond in 1949, will be the subject of this survey.

Here is one classical example of a problem on which this method gives information. It was raised by Gel'fond (p. 127 of the english translation of his book) and by Schneider (7th problem of his book).

Problem of Gel'fond-Schneider. Let α *be a non-zero algebraic number,* $\log\alpha$ *a non-zero determination of its logarithm, and* β *an algebraic number of degree* $d=[\mathbb{Q}(\beta):\mathbb{Q}]$ *with* $d\geqq 2$. *Then the* $d-1$ *numbers*

$$\alpha^{\beta},\ \alpha^{\beta^2},\ \ldots,\ \alpha^{\beta^{d-1}} \tag{$*$}$$

are algebraically independent.

An equivalent formulation of this problem is to ask for the algebraic independence of $\alpha^{\beta_1}, \ldots, \alpha^{\beta_k}$ for algebraic α with $\alpha \neq 0$, $\log\alpha \neq 0$, and for algebraic $\beta_1, \ldots, \beta_k$ with $1, \beta_1, \ldots, \beta_k$ linearly independent over $\mathbb{Q}$.

Hilbert's seventh problem stated that each number in (*) is transcendental, and this was proved by Gel'fond and Schneider in 1934. One main step achieved by Gel'fond in 1948 was to prove that for $d\geqq 3$, at least two of the numbers in (*) are algebraically independent. In particular for $d=2$ and $d=3$ the problem is solved.

Until 1970, Gel'fond's method was restricted to the proof of algebraic independence of two numbers, among certain values of the exponential function (see I §1 below). In the 70's the first works arise which yield the algebraic independence of at least three numbers for the exponential function (I §2), and at the same time the first results appear connected with elliptic functions (I §3).

The next step is to produce fields of large transcendence degree generated by values of the exponential function, or elliptic functions (and more generally by numbers connected with the exponential of an algebraic group). We know essentially three ways of moving in this direction. The first one originates from a preprint of Chudnovsky, Kiev in 1974 (II §1). The second one, which was published very recently, has been developed by Masser and Wüstholz, starting from their joint work on zero estimates on group varieties (II §2). The third one has been initiated by P. Philippon in connection with the elliptic analog of the Gel'fond-Schneider problem, and used by Philippon and Wüstholz for the elliptic analog of the Lindemann-Weierstrass theorem (II §3).

One important tool in most of the previously mentioned works is a transcendence criterion. The first one appears already in the work of Gel'fond in

1949, and has been improved later (III §1). Next we will see a criterion of algebraic independence of Chudnovsky-Reyssat (III §2), and finally another criterion due to Philippon (III §3).

It should be clear, at least at the end of this survey, that the theory is just at the beginning of its life. We will finish by listing a few conjectures in this field (IV).

I Small transcendence degree

In this section, our aim is to produce some fields generated by values of exponential (§1 and §2) or elliptic (§3) functions, for which we know that the transcendence degree is at least two or at least three.

§1 Two algebraically independent values of the exponential function

The first result of algebraic independence produced by A. O. Gel'fond [1948] was the independence of the two numbers α^{β}, α^{β^2} for α algebraic, $\alpha \neq 0$, $\log\alpha \neq 0$, and β cubic. In [1949] and in his book [1], he gave more general results which were extended or refined later by A. A. Shmelev [1967], [1968a], [1968b], R. Tijdeman [1970a], W. D. Brownawell [1969], [1971b], [1971g], [1971i] and others [1971a], [1971f], [1971j], [3] Chap. 7, [4] Chap. 12, [6] Chap. 9 (see also the surveys [1966b], [1974c], and [1979e]).

The following result is now well-known.

Theorem 1. Let $x_1, \ldots, x_m$ be complex numbers which are linearly independent over $\mathbb{Q}$, and $y_1, \ldots, y_l$ be also complex numbers which are linearly independent over $\mathbb{Q}$.

a) *Assume $lm \geqq 2(l+m)$. Then two at least of the lm numbers*

$$e^{x_i y_j}, \qquad (1 \leqq i \leqq m,\ 1 \leqq j \leqq l)$$

are algebraically independent.

b) *Assume $lm \geqq l + 2m$. Then two at least of the $lm + l$ numbers*

$$y_j,\ e^{x_i y_j}, \qquad (1 \leqq i \leqq m,\ 1 \leqq j \leqq l)$$

are algebraically independent.

c) *Assume $lm > l + m$. Then two at least of the $lm + l + m$ numbers*

$$x_i,\ y_j,\ e^{x_i y_j}, \qquad (1 \leqq i \leqq m,\ 1 \leqq j \leqq l)$$

are algebraically independent.

d) *Assume $l=m=2$, and assume also that $e^{x_1y_1}$ and $e^{x_1y_2}$ are algebraic. Then two at least of the six numbers*

$$x_1,\ x_2,\ y_1,\ y_2,\ e^{x_2y_1},\ e^{x_2y_2}$$

are algebraically independent.

Further results, using Baker's method, have been given in [1972c], and by A. A. Shmelev [1973d] and G. V. Chudnovsky [1975c].

A measure of algebraic independence of α^β and α^{β^2} (for β cubic) was derived already in 1950 by Gel'fond and Fel'dman [1950], and improved by W. D. Brownawell [1975i], [1977a].

In [1977a], W. D. Brownawell constructs Liouville numbers a such that for β of degree 3, the three numbers a, a^β, a^{β^2} are algebraically independent (see also [1975b], [1975e] and [1977b]).

Further approximation estimates are due to A. A. Shmelev [1970c], [1973b], [1975g], [1978i], [1983a] p. 166—167.

Related works for the p-adic case have been done by W. W. Adams [1964] and T. N. Shorey [1972b] (see also [1972a]). It is worth it to emphasize here the fact that the p-adic results are still sometimes weaker than their complex analog. For example in the p-adic version of theorem 1 one needs a strict inequality in the assumptions a) and b). In particular for algebraic α and β in $\mathbb{C}_p$ with α neither zero nor root of 1, $|\alpha-1|_p<1$ and $[\mathbb{Q}(\beta):\mathbb{Q}]=2$, with $|\beta^j \log\alpha|_p < p^{-1/(p-1)}$ for $j=1$ and $j=2$, it is not yet proved that α^β and α^{β^2} are algebraically independent (see [3] Appendice).

Existing methods could give the feeling that the problem of algebraic independence of π and e^π should be easier than the problem of the algebraic independence of π and e. Anyway, both problems are still open.

§ 2 *Three algebraically independent values of the exponential function*

The easiest way to give a lower bound for the transcendence degree of some fields has been proposed by S. Lang [1966a], [2] Chap. 5. More or less, the main difficulty which is at the heart of most proofs of algebraic independence is now put in the hypotheses, under the name of "transcendence type". But checking this assumption involves the same kind of difficulties as proving a result of algebraic independence.

One can compare the level of difficulty for giving a measure of algebraic independence of s numbers and that for producing a transcendence degree $\geq s+1$. For instance, transcendence measures are derived in [1949], [1], [1968b], using the method of algebraic independence of two numbers.

On the other hand, if one combines a measure of algebraic independence of s numbers with a method which yields a transcendence degree $\geq t$, one can hope to produce a transcendence degree $\geq t+s$.

For instance, using a transcendence measure, due to A. O. Gel'fond [1] Chap. III § 4, of α^β, W. D. Brownawell [1971b], [1974a] and A. A. Shmelev [1971d] proved the algebraic independence of three at least of the numbers

$$\alpha^\beta, \alpha^{\beta^2}, \ldots, \alpha^{\beta^{d-1}}$$

in the problem of Gel'fond-Schneider, provided that $d=[\mathbb{Q}(\beta):\mathbb{Q}]$ is sufficiently large: $d \geqq 19$ for [1971d], $d \geqq 15$ for [1974a].

After a first attempt by A. A. Shmelev [1971e], G. V. Chudnovsky succeeded to remove this assumption to $d \geqq 7$, by avoiding the use of a transcendence type [1973a]. This method is explained in [1975a], [1976d], and [1979e], and used in [1975b], [1975e], [1975i], [1977a] and [1977b].

From a very general claim by Chudnovsky in [1978g] th. 2 p. 347 (which should be considered as a conjecture) one deduces that the above result should hold also for $d \geqq 5$, and this seems to follow from the arguments given by Chudnovky in [1978h].

It is very likely that one can get the algebraic independence of three numbers with the hypotheses of theorem 1, provided that one replaces the assumptions by:

$$lm \geqq 3(l+m) \quad \text{for a)},$$

$$lm \geqq 2l+3m \quad \text{for b)},$$

and

$$lm > 2(l+m) \quad \text{for c)}.$$

However, for technical reasons, one needs measures of linear independence of $x_1, \ldots, x_m$ and of $y_1, \ldots, y_l$.

A corresponding generalization of part d) of theorem 1 is also possible (see [1978j] for a first step in this direction).

§ 3 Small transcendence degree and elliptic functions

The possibility of producing fields of transcendence degree $\geqq 2$ generated by values of elliptic functions was suggested by S. Lang in [1966a] and [2] Chap. V § 3 (see also A. Altman [1970b] for abelian functions), and a kind of axiomatization was proposed in [1972a], but the first concrete examples were given by W. D. Brownawell and K. K. Kubota [1975d]. Further results are due to A. A. Shmelev [1975f], [1979d], but in all these cases the proven results are very far from the conjectural ones.

The first paper with sharp estimates is Chudnovsky's [1975h], which actually yields the algebraic independence of two numbers: if $\wp$ is an elliptic function with algebraic invariants g_2, g_3, and with complex multiplications in an imaginary quadratic field, each non-zero period ω of $\wp$ is algebraically independent of π. A consequence is the algebraic independence of $\Gamma(1/4)$ and π (the transcendence of $\Gamma(1/4)$ was not yet known!), and also of $\Gamma(1/3)$ and π (see [1975j] and [1976d]). Further results were announced by Chudnovsky at the Oberwolfach conference in 1977 (reference [6] of [1978g]; the lecture was delivered by W. D. Brownawell), and the following result can be proved essentially by the same arguments as [1975h] (see [1978g] th. 4 and [5] Lecture 8).

Theorem 2. Let $\wp$ be a Weierstrass elliptic function with algebraic invariants g_2, g_3, let ω be a non-zero period of $\wp$, η be the associated quasi-period of the Weierstrass zeta function:

$$\zeta(z+\omega)=\zeta(z)+\eta,$$

and u a complex number, which is not a pole of $\wp$, with u and ω linearly independent over $\mathbb{Q}$, and $\wp(u)$ algebraic. Then the two numbers

$$\zeta(u)-\frac{\eta}{\omega}u,\ \frac{\eta}{\omega}$$

are algebraically independent.

Several consequences of this result on the modular function and its derivatives have been given by D. Bertrand in [1978c], together with a p-adic analog. Bertrand pursued these investigations in [1978d] and [1978e] where he gave results of algebraic independence of the values of modular functions. (See also Ianchenko, [1983a] p. 174—175.)

At Helsinki [1978g], G. V. Chudnovsky announced further results (esp. th. 5 and 6 p. 344, and th. 4 p. 347), and at the same time he produced lists of similar statements (e. g. reference [7] of [1978g]; here we decided not to include unpublished manuscripts in our bibliography). The proofs of these statements are not published, but it is now a routine matter to check these results (and it has been done for the most interesting of them).

Concerning this subject we mention two papers by N. I. Fel'dman [1979c], [1980b], (cf. also [1983a] p. 148) and a paper by P. Philippon [1979f] where he introduces his "fausses variables" which will play an important role in 1982.

In the above mentioned meeting of Oberwolfach in March 1977 (which is referred to also p. 270 of [1979a]), G. V. Chudnovsky explained how Gel'fond's method for the algebraic independence of two numbers can be used to yield the case $n=2$ of the Lindemann-Weierstrass theorem. From [1979a] § 3, using the arguments of [1978h], the same is true for $n=3$. Then, in [1979a], he extends his method to the elliptic case (thanks to a subtle but technical device known as the Baker-Coates-Anderson lemma):

if $\wp$ has algebraic invariants g_2, g_3, and if $\alpha_1, \ldots, \alpha_5$ are algebraic numbers linearly independent over $\mathbb{Q}$, then among $\wp(\alpha_1), \ldots, \wp(\alpha_5)$, there are at least three algebraically independent numbers.

In the case of complex multiplication, this yields the algebraic independence of $\wp(\alpha_1)$, $\wp(\alpha_2)$, $\wp(\alpha_3)$ when $\alpha_1, \alpha_2, \alpha_3$ are linearly independent over the ring of endomorphisms.

Better results are announced in:
[1979b], th. 2.3 and 2.4: algebraic independence of four numbers
[1978g], th. 1 p. 348, and [1979a] §6: algebraic independence of six numbers
[1980d], th. 10.4 p. 71: algebraic independence of n numbers (with a measure!), but proofs are not given (see II §3 below).

The main difficulty in the proofs for the small transcendence degrees is to exhibit a non-zero value of the auxiliary function. Fundamental progress on this matter have been achieved in the last ten years, first by D. W. Masser, then by W. D. Brownawell and D. W. Masser, and later by D. W. Masser and G. Wüstholz (see the address [1983c] of D. W. Masser at the international congress of Warsaw).

Using the method of zero-estimates of Brownawell-Masser, G. Wüstholz [1979g] gave a general statement (in the style of the Schneider-Lang theorem) which yields the algebraic independence of two numbers among values of functions satisfying certain differential equations. In concrete examples this result gives rather crude estimates, but these can be improved as shown by E. Reyssat [1980a], [1980f]. One motivation for these works is the unsolved problem of the algebraic independence of the three numbers π, e^π, $\Gamma(1/4)$; [1980f] contains the best known partial results in this direction, involving elliptic integrals of second and third kind, and transcendence degree at least two or three.

The elliptic analog of theorem 1 has been obtained in [1980c] by Masser and Wüstholz as a direct consequence of their works on zero estimates on algebraic groups. For instance when $\wp$ has algebraic g_2, g_3 and complex multiplication by an imaginary quadratic field k, when u is a non-zero complex number such that $\wp(u)$ is defined and algebraic, and when β is algebraic of degree 3 over k, then the two numbers $\wp(\beta u)$ and $\wp(\beta^2 u)$ are algebraically independent. In [1982i], R. Tubbs gives a measure of algebraic independence for these two numbers.

Finally general results can now be obtained involving n-parameter subgroups of algebraic groups [1981d] §5, [1982h].

II Large transcendence degree

We begin with a method of Chudnovsky (§1), which works only for exponential functions, and yields rather weak estimates. Then we mention a method of Masser and Wüstholz (§2), which gave the first results for elliptic functions, with estimates of essentially the same quality. Fortunately, the third method (§ 3) gives sometimes best possible results.

§ 1 A method of Chudnovsky and related works

During his "Tagung über diophantische Approximationen" at Oberwolfach in July 1974, Th. Schneider received from G. V. Choodnovsky the canonical sheet (see the copy on the next page).

The first theorem was also stated in [1973a] (p. 398 in the english translation), [1974b] (with measures of algebraic independence in th. 4), and [1974d] th. 1.1. A sketch of the proof can be found in [1976d] and [1979e].

The only place where Chudnovsky provides a proof for large transcendence degree is the Kiev preprints [1974e]. It is difficult to answer the question of whether or not the proof given there is complete. The arguments are quite convoluted, and attempts to write down all the details involved non-trivial problems. The questions (especially from Warkentin) raised to Chudnovsky at Oberwolfach in May—June 1979 were not answered.

On the other hand there is no doubt that the method of [1974e] leads to fields of large transcendence degree generated by values of the exponential function. This has been checked by P. Warkentin in [1978f]. The proof of Philippon [1981a] and Reyssat [1981b] has been inspired by Chudnovsky's one, but the details are comparatively extremely easy. Further works on this subject, due to R. Endell [1981c], have been completed by W. D. Brownawell.

A rather different approach, due to Ju. V. Nesterenko [1982f], yields the same result.

Theorem 3. Let α and β be algebraic numbers, $\alpha \neq 0$, $\log\alpha \neq 0$, and let $d = [\mathbb{Q}(\beta):\mathbb{Q}]$ satisfy $d \geq 2$. Then the transcendence degree over $\mathbb{Q}$ of the field generated by

$$\alpha^{\beta}, \alpha^{\beta^2}, \ldots, \alpha^{\beta^{d-1}}$$

is at least $\left[\dfrac{\log(d+1)}{\log 2}\right]$.

(The brackets denote the integral part.)

In the p-adic case, the transcendence degree is at least $\left[\dfrac{\log d}{\log 2}\right]$ as shown by Philippon in [1981a].

A quantitative refinement of theorem 3 is given in [1982f] th. 2 (compare with th. 4 of [1974b]). For the proof of this estimate, Nesterenko develops the methods of commutative algebra which he introduced in the subject for estimating the multiplicity of zeros. (See also [1983a] p. 102—104.)

Mathematisches Forschungsinstitut
Oberwolfach

Tagung:

Vortragender: G.V. Choodnovsky (Kiev, USSR)

Thema des Vortrages: Several questions in the theory of transcendental and algebraically independent numbers

Vortragsdauer:

Kurze Zusammenfassung: (höchstens 15 Zeilen)

This work is devoted to the basic investigation of the arithmetic nature of numbers, connected with the values of exponent. Let's denote by M, N and n any natural numbers ($\geqslant 1$) and by $\alpha_1, \ldots, \alpha_N$, $\beta_1, \ldots, \beta_M$ linearly independent over Q sequences of complex numbers. We choose three sets of numbers: $\overline{S}_1 = \{e^{\alpha_i \beta_j}\}$, $\overline{S}_2 = \{\beta_j, e^{\alpha_i \beta_j}\}$, $\overline{S}_3 = \{\alpha_i, \beta_j, e^{\alpha_i \beta_j}\}$, $1 \leqslant i \leqslant N$, $1 \leqslant j \leqslant M$ and such constants $\varkappa_1 = MN/(M+N)$, $\varkappa_2 = M(N+1)/(M+N)$, $\varkappa_3 = \varkappa_1 + 1$.

Theorem 1. If $\varkappa_i \geqslant 2^n$ ($\varkappa_3 > 2^n$ for $i = 3$), then in the set $\overline{S}_i$ there exist $n+1$ algebraically independent numbers.

We obtain a good estimation of measure of transcendence in the case of bounded height:

Theorem 3. For $P(x) \in Z[x]$, $P(x) \not\equiv 0$ of degree $\leqslant d$ and height $\leqslant H$ we have

$$|P(e^{\pi})| > \exp(-c_1 d \ln H \ln^2(d \ln H));$$

$$|P(e)| > \exp(-c_2 d^2 \ln(dH)).$$

Wir bitten, diesen Vortragsauszug an die Geschäftsstelle des Mathematischen Forschungsinstituts in 78 Freiburg i. Br., Albertstr. 24, einzusenden oder ihn spätestens bei Beginn der Tagung dem Tagungsleiter zu übergeben.

The conclusion of theorem 3 can be written $2^t > (d+1)/2$, where t is the transcendence degree. The right hand side $(d+1)/2$ is, to a certain extent, the limit in the present stage of Gel'fond's method. The left hand side 2^t arises from the criterion of algebraic independence or the induction procedure, and one should be able to replace it by $t+1$. The conclusion would be $t \geqq [(d+1)/2]$, which is "half" of the Gel'fond-Schneider problem. This result was claimed first by Gel'fond in [1948] p. 280, then by Chudnovsky (th. 2 p. 347 of his Helsinki adress [1978g]; see also p. 288—290 of [1976d] which is taken from a letter of Chudnovsky dated May 1976). The only information given by Chudnovsky on his proof is that it uses the resolution of singularities ([1978h] p. 1—2, [1979a] p. 267, [1980d] p. 50).

He tried also to generalize his method to elliptic functions. In [1979a] p. 268 he agrees that he did not succeed to solve the technical difficulties, but in [1980d] (p. 15 and p. 70—71) he states general results (even with a measure of algebraic independence). Overly optimistic statements can be found in [1980e] (see Masser's comments in Zbl. 456.10016).

Apart from his Kiev preprints [1974e], all the proofs provided by Chudnovsky so far deal only with small transcendence degrees.

§ 2 A method of Masser and Wüstholz

In their joint paper [1982d], Masser and Wüstholz develop a new method for obtaining results of algebraic independence. Here is the concrete case they work out.

Theorem 4. Let $\wp$ be an elliptic function with algebraic invariants g_2, g_3, and without complex multiplication. Let $x_1, \ldots, x_d, y_1, \ldots, y_l$ be complex numbers. Assume that $\varkappa > 0$ is such that, for sufficiently large A, B, and for all integers $(a_1, \ldots, a_d, b_1, \ldots, b_l)$ with $\max|a_i| = A$ and $\max|b_j| = B$, we have

$$|a_1 x_1 + \ldots + a_d x_d| \geqq \exp(-A^\varkappa)$$

and

$$|b_1 y_1 + \ldots + b_l y_l| \geqq \exp(-B^\varkappa).$$

Let K be the field generated by the values of $\wp$ at the points $x_i y_j$, $(1 \leqq i \leqq d$, $1 \leqq j \leqq l)$ where $\wp$ is defined, and let t be the transcendence degree of K over $\mathbb{Q}$. Then

$$2^{t+2}(t+8) + 4\varkappa > ld/(l+2d).$$

This is the first result giving many algebraically independent values of elliptic functions.

The starting point is an improved version of their zero estimate. The second main tool is an effective version of the Hilbert Nullstellensatz. Next they need an explicit description of all algebraic subgroups of a power of an elliptic curve, and for that they have to improve a result of Kolchin.

This method involves a rather heavy machinery, and it is not clear whether it can be extended to a general commutative algebraic group (defined over a finitely generated extension of $\mathbb{Q}$).

The final lower bounds are the logarithms of that one expects. However this is due only to the weak estimate in the effective Nullstellensatz. An important problem is to improve this estimate.

§ 3 A method of Philippon and related works

This method arised in connection with the elliptic analog of Gel'fond-Schneider's problem. One main idea [1982a] is to use the action of certain endomorphisms on an algebraic group (in a different situation, Bertrand and Masser had used similar actions to deduce results of linear independence from the theorem of Schneider-Lang). Then Philippon uses a transcendence criterion of his own (see III §3 below). The following statement of his thesis [1983b] improves his earlier results of [1982a] and [1982b].

Theorem 5. Let $\wp$ be an elliptic function with algebraic g_2, g_3, k be the field of endomorphisms of the elliptic curve, u a non-zero complex number, and β an algebraic number of degree $d \geqq 2$ over k. Assume that $u, \beta u, \ldots, \beta^{d-1} u$ are not poles of $\wp$. Then the transcendence degree t over $\mathbb{Q}$ of the field generated by

$$\wp(u),\ \wp(\beta u),\ \ldots,\ \wp(\beta^{d-1} u)$$

satisfies

$$t \geqq [(d-1)/3] \quad \text{if} \quad k = \mathbb{Q}$$

and

$$t \geqq [(d-1)/2] \quad \text{if} \quad k \neq \mathbb{Q}.$$

One expects $t \geqq d-1$ in both cases. It is remarquable that the elliptic result (th. 5) is stronger than its exponential analog (th. 3).

The results of [1982a], [1982b] and [1983b] are more general, dealing with abelian varieties, and the action of certain endomorphisms.

The same kind of action occurs also in the papers [1982c], [1982g] of Wüstholz, and [1982e] of Philippon, connected with Chudnovsky's method of [1979a]. We do not quote the general statements, but only the following striking consequence:

Theorem 6. Let $\wp$ be an elliptic function with algebraic invariants g_2, g_3, and with complex multiplications in an imaginary quadratic field k. Let $\alpha_1, \ldots, \alpha_n$ be algebraic numbers which are linearly independent over k. Then the n numbers

$$\wp(\alpha_1), \ldots, \wp(\alpha_n)$$

are algebraically independent.

The method is extended in [1982h] and [1983b] to n-parameter subgroups of an abelian variety.

A difficult problem here is to extend the method for the study of non-complete group varieties. The difficulty arises from the use of Philippon's criterion or the elimination procedure.

Another limitation of the method is due to the hypotheses on the endomorphisms which do not enable one to study general situations like in th. 1 and th. 4.

III Criteria of algebraic independence

One of the basic tools in Gel'fond's method is a criterion which replaces the fundamental Liouville estimate (size inequality) of transcendence proofs. We consider first the criteria in one or two variables (§ 1), then an extension to t variables (§ 2). The method, by induction, yields an exponent 2^t in general. A different approach, based on Nesterenko's work on commutative algebra, enables Philippon to get another criterion with the right exponent $t+1$.

§ 1 Gel'fond's criterion, and first generalizations

a) *One variable over* $\mathbb{Q}$

Gel'fond's criterion appears first in [1949] and in his book [1] (Chap. III §4 lemma 7). Refinements are given in [1965] §6, [1970a] lemmas 6 and 6', [1971a] §3, [1971h], [1976c] critère 2.4 et remarque 2.10. See also [3] Chap. 5, [4] Chap. 12 §4, [5] §8.2, [6] lemma 3.9, and the expositions of Brownawell [1979e] and Chudnovsky [1979h].

For $P \in \mathbb{C}[X]$, $P \neq 0$, we write $\deg P$ for its degree, and $H(P)$ for the maximal absolute value of its coefficients.

Theorem 7. Let θ be a complex number, $(t_N)_{N\geq 0}$ and $(d_N)_{N\geq 0}$ two sequences of numbers $\geqq 1$, and $(P_N)_{N\geq 1}$ a sequence of non-zero elements of $\mathbb{Z}[X]$. We define

$$T_N = 2(d_{N-1} + d_N + d_{N+1})(t_{N-1} + t_N + t_{N+1}), \quad (N \geqq 1),$$

and we assume $\lim T_N = +\infty$ *as* $N \to +\infty$. *We assume further that for all* $N \geqq 1$,

$$\deg P_N \leqq d_N, \quad \deg P_N + \log H(P_N) \leqq t_N,$$

and

$$|P_N(\theta)| \leqq e^{-T_N}.$$

Then θ is algebraic and $P_N(\theta)=0$ for all $N \geqq 1$.

Here is a sketch of the proof. Since $|P_N(\theta)|$ is small, θ is close to a root α_N of P_N. Let s_N be its multiplicity:

$$|\theta - \alpha_N|^{s_N} \leqq e^{-T'_N}, \quad \text{where} \quad T'_N = T_N - 2d_N t_N.$$

Define $d'_N = d_N/s_N$, $t'_N = t_N/s_N$. It is readily verified that

$$|\alpha_N - \alpha_{N+1}| \leqq 2\exp(-d'_N d'_{N+1} - t'_N d'_{N+1} - t'_{N+1} d'_N).$$

On the other hand Liouville's inequality shows that this estimate cannot hold unless $\alpha_N = \alpha_{N+1}$. The theorem follows at once.

b) *One variable with finite transcendence type*

The preceeding proof works as well if one replaces the rational field by a finitely generated extension of $\mathbb{Q}$ on which one assumes some type of Liouville inequality (Lang's transcendence type [2] Chap. V). This was worked out by W. D. Brownawell [1969], [1974a] and A. A. Shmelev [1973c]. A further result was obtained by Chudnovsky [1976a] using a rather subtle argument (coloured sequences; see also [1975a], [1976c], [1978h] Coroll. 5.4, [1979e], [1979h]).

c) *Two variables*

From a well-known example due to Cassels (see for instance [1971c] p. 670), the obvious generalization of Gel'fond's criterion to several variables ([1965] p. 191—192) does not hold without further assumptions.

The first generalization of this type was given already for t variables (see § 2 below). However it is worthwhile mentionning that better results are available for two variables. The two papers [1978h] and [1980d] by Chudnovsky contain description of methods which lead to sharp results for the algebraic independence of three numbers, and his paper [1979h] contains explicitly criteria of algebraic independence in two variables ([1979h], th. 4.1 p. 339—340; see also [1980c], Prop. 1 and 13).

§ 2 A criterion of Chudnovsky-Reyssat

The Kiev preprints [1974e] contain a description of Chudnovsky's method for algebraic independence of t numbers, with some details. They do not contain an explicit criterion (the quotation in [1979h] p. 326 is misleading). Such a criterion is stated without proof in [1974d] lemma 1.2 p. 23 (probably the conclusion should be algebraically dependent, rather than algebraically independent, but anyway one can show [1983d] that the hypotheses are never satisfied—thus the result is true!).

The idea of Chudnovsky is to assume not only an upper bound for the values of the polynomials, but also a lower bound. In the transcendence proof this new hypothesis is checked by using a "small value lemma" for exponential polynomials, due to R. Tijdeman.

A more general criterion has been stated—and proved— by E. Reyssat in [1981b]. The proof uses several ideas of Chudnovsky in [1974c], and especially the semi-resultant (see also [1976b] and [1979e]). The induction performed by Reyssat in [1981b] is rather more tricky than it appears. See also [1983d].

Unfortunately the induction procedure leads to an exponent 2^t where one expects $t+1$. There are claims by Chudnovsky of a criterion with the right exponent (e.g. [1974d] p. 30, [1979h] Prop. 5.1 p. 357—358). However they are not supported by proofs, so far, and Philippon noticed that Cassel's construction leads to counter-examples to the statements at the end of § 5 of [1979h] p. 359.

§ 3 *A criterion of Philippon*

A generalization of Gel'fond's criterion to several variables was proposed by R. Dvornicich in [1978a]. He replaces the sequence of polynomials by a sequence of ideals in a polynomial ring. However for the proof he had to assume that the ideals are prime, and this hypothesis is too strong for applications.

Using methods of commutative algebra (see also [1982f]), Philippon succeeded in proving a very good criterion ([1982a] Prop. 1, [1982b] Th. 1.4, and [1983b] Chap. 1), which he used for his proof of theorem 5 (see II § 3), and which will certainly have other applications later.

For simplicity we quote only a special case of a criterion in [1983a].

Theorem 8. For an integer $n \geqq 1$, let $\theta \in \mathbb{P}_n(\mathbb{C})$, let $(\theta_0, \ldots, \theta_n) \in \mathbb{C}^{n+1}$ be projective coordinates of θ, and $\mathscr{E}$ a prime homogeneous ideal of the ring $A = \mathbb{Q}[X_0, \ldots, X_n]$, of codimension $n-t$, such that $e(\theta) = 0$ for all $e \in \mathscr{E}$. Let a be a real number, $a > 1$.

There exists a constant $C > 0$ with the following property. Let $(t_N)_{N \geq 1}$, $(d_N)_{N \geq 1}$ be two sequences of numbers $\geqq 1$, with

$$d_N \leqq d_{N+1} \leqq a d_N, \quad t_N \leqq t_{N+1} \leqq a t_N \quad \textit{for all} \quad N \geqq 1,$$

and

$$\lim d_N = +\infty \quad as \quad N \to +\infty.$$

Let $(I_N)_{N\geq 1}$ be a sequence of homogeneous ideals of A of codimension $\geqq n$; assume that for each $N \geqq 1$ there exist an integer $m(N) \geqq 1$ and homogeneous polynomials $Q_1^{(N)}, \ldots, Q_{m(N)}^{(N)}$ in $\mathbb{Z}[X_0, \ldots, X_n]$ with

$$I_N = (\mathscr{E}, Q_1^{(N)}, \ldots, Q_{m(N)}^{(N)}),$$
$$\deg Q_j^{(N)} + \log H(Q_j^{(N)}) \leqq t_N, \ \deg Q_j^{(N)} \leqq d_N,$$

and

$$|Q_j^{(N)}(\theta_0, \ldots, \theta_n)| \leqq \exp(-C t_N d_N^t),$$

for $1 \leqq j \leqq m(N)$. Then for all sufficiently large N, θ is a zero of I_N. In particular $\theta \in \mathbb{P}_n(\bar{\mathbb{Q}})$, where $\bar{\mathbb{Q}}$ is the field of algebraic numbers.

When infinitely many of the ideals I_N are of codimension $n+1$, Philippon uses an effective elimination procedure (see e.g. [1982b] th. 1.3). For this special case (which is sufficient in the proof of theorem 6, II §3 above), Wüstholz uses the U-resultant of Macaulay [1982g].

The method of Masser-Wüstholz in [1982d] (see II §2) does not use a transcendence criterion explicitly, but the elimination is performed through the use of Hilbert's Nullstellensatz.

IV Further conjectures

We consider first the usual exponential function, then elliptic functions, and finally algebraic groups.

§ 1 Exponential function

A very general conjecture concerning the transcendence and algebraic independence properties of the exponential function has been made by S. Schanuel [2] p. 30.

Schanuel's conjecture. Let $x_1, \ldots, x_n$ be complex numbers which are linearly independent over $\mathbb{Q}$. Then the transcendence degree of the field

$$\mathbb{Q}(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n})$$

over $\mathbb{Q}$ is at least n.

Most statements (either proved or conjectural ones) concerning algebraic independence of numbers connected with exponential or logarithms are consequences of this conjecture (see [3] §7.5 + Exercices 7.5 a, ..., e). This is the case for instance of the Gel'fond-Schneider problem stated in the introduction, and of the problem of the algebraic independence of $\log\alpha_1, \ldots, \log\alpha_n$, when $\alpha_1, \ldots, \alpha_n$ are non-zero algebraic numbers and $\log\alpha_1, \ldots, \log\alpha_n$ are $\mathbb{Q}$-linearly independent ([1] p. 127 and p. 177, [2] p. 31, [3] conj. 7.5.3, [4] p. 119—120).

The p-adic analog of Schanuel's conjecture is stated in [1981a]. An open problem is to prove a p-adic version of the Lindemann-Weierstrass theorem (see [3] p. A9).

Finally, let us try a first generalization of Schanuel's conjecture in the realm of diophantine approximations.

Let $x_1, \ldots, x_n$ be complex numbers which are linearly independent over $\mathbb{Q}$. Let d be a positive integer. Then there exists a positive number $C = C(x_1, \ldots, x_n, d)$ with the following property: for all $P_1, \ldots, P_{n+1}$ in $\mathbb{Z}[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$ of degrees $\leq d$ and heights $\leq H_1, \ldots, H_{n+1}$, which generate an ideal of $\mathbb{Q}[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$ of rank $n+1$, we have

$$\sum_{j=1}^{n+1} |P_j(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n})| \cdot H_j^C \geq 1/C.$$

§ 2 *Elliptic functions*

Let $\wp$ be an elliptic function of Weierstrass. We denote by g_2, g_3 the invariants, ω_1, ω_2 a pair of fundamental periods, and η_1, η_2 the associated quasiperiods. According to Chudnovsky [1978g] [4] p. 343, the transcendence degree of the field $\mathbb{Q}(g_2, g_3, \omega_1, \omega_2, \eta_1, \eta_2)$ over $\mathbb{Q}$ is at least 2. It can be as small as 2 (when $\wp$ has complex multiplications). It would be interesting to decide whether this bound can be improved when there is no complex multiplication: in this case, for g_2 and g_3 algebraic, is it true that the four numbers $\omega_1, \omega_2, \eta_1, \eta_2$ are algebraically independent?

Another problem is to list the algebraic relations between the periods and quasi-periods of several $\wp$ functions. For instance one would like to know that the three numbers π, $\Gamma(1/4)$, $\Gamma(1/3)$ are algebraically independent.

The analog of the Lindemann-Weierstrass theorem for elliptic functions without complex multiplication is not yet known. One would like to prove at least "half" of it: the transcendence degree over $\mathbb{Q}$ of $\mathbb{Q}(\wp(\alpha_1), \ldots, \wp(\alpha_n))$ is at least $n/2$ (for $\alpha_1, \ldots, \alpha_n$ algebraic numbers $\mathbb{Q}$-linearly independent). Philippon [1983b] proves this result under the additional assumption $\mathbb{Q}(\alpha_1, \ldots, \alpha_n) = \mathbb{Q}\alpha_1 + \ldots + \mathbb{Q}\alpha_n$.

§ 3 *Algebraic groups*

A proof of the transcendence of $\Gamma(1/5)$ (and other values of the gamma function) would follow from a complete description of the algebraic relations between the coordinates of the periods of abelian functions (see [2] Chap. IV, historical note, [1971c], and the papers by Deligne, Ribet and Shimura in "Fonctions Abéliennes et Nombres Transcendants", Mémoire S. M. F. n°2 [1980]).

The following problem is proposed by Philippon [1983b] as a generalization of the Lindemann-Weierstrass theorem: *let G be an algebraic group which is defined over the field $\bar{\mathbb{Q}}$ of algebraic numbers, $\alpha \in T_G(\bar{\mathbb{Q}})$, X the Zariski closure of* $\exp_G \alpha$ *over* $\bar{\mathbb{Q}}$, *and H the smallest algebraic subgroup of G containing* $\exp_G \alpha$. *Assuming* $\mathrm{Hom}_{\bar{\mathbb{Q}}}(H, \mathbb{G}_a) = 0$, *is it true that X is a connected component of H?*

As a consequence, if A is a simple abelian variety of dimension d (without assumptions on its endomorphisms), for $\alpha \in T_A(\bar{\mathbb{Q}})$, $\alpha \neq 0$, the point $\exp_G \alpha$ has a transcendence degree $\geq d$ (see [2] p. 42).

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