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Department of Mathematics
Pondicherry University

An introduction to Diophantine analysis and transcendental number theory

Michel Waldschmidt

Institut de Mathématiques de Jussieu & CIMPA
http ://www.math.jussieu.fr/∼miw/

An introduction to Diophantine analysis and transcendental number theory

A (real or complex) number is either rational (root of a non-zero linear polynomial $aX + b$ with rational coefficients), or algebraic irrational (root of a non-linear polynomial with rational coefficients), or else transcendental. For a number which is given by an analytic formula (limit of a sequence, a series or a product, value of an integral or a function), it is most often very hard to tell in which of the three above-mentioned subsets it falls. The main tool to attack this question is to investigate its approximation properties by rational or algebraic numbers.

We survey some of the main results related to these questions, old as well as recent ones, we give some applications and we quote a few open problems.

An introduction to Diophantine analysis and transcendental number theory

Michel Waldschmidt

1. Diophantine equations
2. Diophantine Approximation
3. Irrationality
4. Transcendence

http ://www.math.jussieu.fr/~miw/

Diophantine equations

- Example : Fermat's equation $x^n + y^n = z^n$
  n is fixed, the unknowns are x, y and z in $\mathbf{Z}$.
- More generally : fix a polynomial F in m variables and consider the equation

$$F(x_1, \dots, x_m) = 0$$

  where the unknown $x_1, \dots, x_m$ are either rational integers or rational numbers.
- *Exponential Diophantine equations* : when some exponents are also unknown.
  Pillai's Conjecture (1945) : Let k be a positive integer. The equation

$$x^p - y^q = k,$$

  where the unknowns x, y, p and q take integer values, all ≥ 2, has only finitely many solutions (x, y, p, q).

The Ramanujan–Nagell equation

- Ramanujan : the equation

$$x^2 + 7 = 2^n$$

has the solutions

$$\begin{array}{rcccr} 1^2 + 7 & = & 2^3 & = & 8 \\ 3^2 + 7 & = & 2^4 & = & 16 \\ 5^2 + 7 & = & 2^5 & = & 32 \\ 11^2 + 7 & = & 2^7 & = & 128 \\ 181^2 + 7 & = & 2^{15} & = & 32\,768 \end{array}$$

- Nagell (1948) : no further solution

Historical survey

- Pierre de Fermat (1601 - 1665)
- Leonhard Euler (1707 - 1783)
- Joseph Louis Lagrange (1736 - 1813)
- XIXth Century : Hurwitz, Poincaré
- Mordell's Conjecture : rational points
- Siegel's Theorem (1929) : integral points
- Faltings' Theorem(1983) : finiteness of rational points on an algebraic curve of genus ≥ 2 over a number field.
- Andrew Wiles (1993) : proof of Fermat's last Theorem

$$a^n + b^n = c^n \qquad (n \geq 3)$$

History of rational approximation theory

- Diophantine approximation is the study of the approximation of a real or complex number by rational or algebraic numbers.
- It has its early sources in astronomy, with the study of movement of the celestials bodies, and in the computations of π.

Diophantine approximation in the real life

Small divisors and dynamical systems (H. Poincaré)
Periods of Saturn orbits (Cassini divisions)
Stability of the solar system
Resonance in astronomy
Engrenages
Quasi-cristals
Acoustic of concert halls
Calendars : bissextile years

Computation of π

- Rhind Papyrus : $2^5/3^4 = 3.1604\ldots$
- Baudhāyana (Sulvasūtras) : $3,088$
- Suryaprajnapati (Jaina mathematician) : $\sqrt{10} = 3.162\ldots$
- Archimedes : 3.1418
- Chan Hong Wang Fan, Liu Hui, Zu Chongzhi (Tsu Ch'ung-Chih) : $355/113 = 3.1415929\ldots$
- Aryabhaṭīya, Āryabhaṭa I : $3,1416$ (suggests $\pi \notin \mathbf{Q}$)
- Bhāskara I : suggests a negative solution to the problem of squaring the circle.
- Bhāskarācārya (Bhāskara II) : $3927/1250 = 3,1416\ldots$
- Madhava (1380–1420) : series, 11 exact decimals 3.14159265359 (Viète 1579 : 9 decimals only).

Rational approximation to real numbers

- The set $\mathbf{Q}$ of rational numbers is dense in the set $\mathbf{R}$ of real numbers : *for any real number* ξ *and any* $\epsilon > 0$ *there exists* $p/q \in \mathbf{Q}$ *such that*

$$\left|\xi - \frac{p}{q}\right| < \epsilon.$$

- Better estimate : for $\xi \in \mathbf{R}$ and $q \geq 1$ consider the nearest integer p to $q\xi$. Then $|q\xi - p| \leq 1/2$, hence

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{2q}.$$

Rational Diophantine approximation

- For computing a number with a sharp accuracy, one wishes to get many decimals (or binary digits) with a small number of operations (products, say).
- For Diophantine questions, the cost is measured by the denominator q : one investigates how well ξ can be approximated in terms of q.
- A rational number has a single good approximation, itself ! Indeed *if* $\xi = a/b$ *is a given rational number, then for any* $p/q \in \mathbf{Q}$ *distinct from* ξ,

$$\left|\xi - \frac{p}{q}\right| \geq \frac{c}{q}$$

where $c = 1/b$. **Proof** : $|bq - ap| \geq 1$.

Rational approximation to an irrational number

- On the opposite, a real irrational number ξ has very sharp rational approximations : *there exist infinitely many* p/q *for which*

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^2}.$$

- Hence to prove that a real number is irrational, it suffices to produce rational approximations p/q better than ϵ/q, but in fact there exist much better approximations, namely in $1/q^2$.
- So it should not be so difficult to prove that a given number is irrational ?

Open problems

- Is Euler's constant

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right) = 0.5772157\ldots$$

irrational ?
- Is $\zeta(5)$ irrational ? (*For* $\zeta(3)$ *: Apéry 1978*).

$$\zeta(s) = \sum_{n\geq 1}\frac{1}{n^s}$$

- Is $e+\pi$ irrational ?

Open problems (continued)

- Is $\Gamma(1/5)$ irrational ?

$$\Gamma(z) = e^{-\gamma z}z^{-1}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1}e^{z/n}.$$

$$\Gamma(s) = \int_0^{\infty} e^{-t}t^s\cdot\frac{dt}{t},$$

$$\mathrm{B}(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 x^{a-1}(1-x)^{b-1}dx$$

- $$\pi = \Gamma(1/2)^2 = \int_0^1 x^{-1/2}(1-x)^{-1/2}dx.$$

$$\int_0^1 \frac{dx}{\sqrt{x-x^3}} = \frac{1}{2}\mathrm{B}(1/4,1/2) = \frac{\Gamma(1/4)^2}{2^{3/2}\pi^{1/2}}$$

$$\int_0^1 \frac{dx}{\sqrt{1-x^3}} = \frac{1}{3}\mathrm{B}(1/3,1/2) = \frac{\Gamma(1/3)^3}{2^{4/3}3^{1/2}\pi}$$

Existence of rational approximations to a real irrational number

Let ξ be a real number.

- From Dirichlet's box principle one deduces that *for each* $Q > 1$ *there exists* $p/q \in \mathbf{Q}$ *with* $1 \leq q < Q$ *such that*

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{qQ}.$$

- It follows that if ξ is irrational, then *there are infinitely many* $p/q \in \mathbf{Q}$ *satisfying*

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^2}.$$

Construction of rational approximations

- Let $x > 1$ be a real number. Consider a rectangle with sides 1 and x.
- Fill it as much as possible with squares of sides 1.
- A small rectangle remains.
- Fill the small rectangle with squares, as much as possible.
- Repeat the process.

Construction of rational approximations : reverse the process

- Let $x > 1$ be a real number.
- Start with the right number of smallest squares, but ignore the remaining small rectangle. Say the small square has sides 1.
- Put on top of these small squares the right number of large squares.
- Continue up to having a rectangle with sides p and q where p/q is a rational number very close to x.
- For instance starting with a single small square and placing each time a single square we produce the sequence of Fibonacci numbers $F_0 = 1$, $F_1 = 1$, $F_2 = 2$, $F_3 = 3$, $F_{n+1} = F_n + F_{n-1}$.

The algorithm of continued fractions

Let $x \in \mathbf{R}$.

- Write
$$x = [x] + \{x\} \quad \text{with } [x] \in \mathbf{Z} \text{ and } 0 \le \{x\} < 1.$$
- If x is not an integer, then $\{x\} \neq 0$ and we set $x_1 = 1/\{x\}$, so that
$$x = [x] + \frac{1}{x_1} \quad \text{with } [x] \in \mathbf{Z} \text{ and } x_1 > 1.$$
- If x_1 is not an integer, we set $x_2 = 1/\{x_1\}$:
$$x = [x] + \cfrac{1}{[x_1] + \cfrac{1}{x_2}} \quad \text{with } x_2 > 1.$$

Continued fraction expansion

Set $a_0 = [x]$ and $a_i = [x_i]$ for $i \geq 1$.

- Then :

$$x = [x] + \cfrac{1}{[x_1] + \cfrac{1}{[x_2] + \cfrac{1}{\ddots}}} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

  the algorithm stops after finitely many steps if and only if x is rational.

- We use the notation

$$x = [a_0 \,;\, a_1,\ a_2,\ a_3 \ldots]$$

- Remark : if $a_k \geq 2$, then $[a_0 \,;\, a_1,\ a_2,\ a_3, \ldots, a_k] = [a_0 \,;\, a_1,\ a_2,\ a_3, \ldots, a_k - 1, 1]$.

Continued fractions and rational approximation

For

$$x = [a_0;\ a_1,\ a_2, \ldots, a_k, \ldots]$$

the sequence of rational numbers

$$p_k/q_k = [a_0;\ a_1,\ a_2, \ldots, a_k] \qquad (k = 1, 2, \ldots)$$

give rational approximations for x which are the best ones when comparing the quality of the approximation and the size of the denominator.

Continued fractions : examples

- The developments

$$[1], \quad [1;1], \quad [1;1,1], \quad [1;1,1,1], \quad [1;1,1,1,1], \quad [1;1,1,1,1,1]\ldots$$

are the quotients

$$\begin{array}{cccccc} F_2/F_1 & F_3/F_2 & F_4/F_3 & F_5/F_4 & F_6/F_5 & F_7/F_6 \\ \| & \| & \| & \| & \| & \| & \ldots \\ 1 & 2 & 3/2 & 5/3 & 8/5 & 13/8 \end{array}$$

of consecutive Fibonacci numbers

$$(F_n)_{n\geq 0} = 0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ \ldots$$

- The development $[1;1,1,1,1\ldots]$ is the continued fraction expansion of the Golden Number

$$\Phi = \frac{1+\sqrt{5}}{2} = \lim_{n\to\infty} \frac{F_{n+1}}{F_n} = 1,6180339887499\ldots$$

which satisfies

$$\Phi = 1 + \frac{1}{\Phi}\cdot$$

Further examples

- The continued fraction expansion of the number $\sqrt{2} = 1,4142135623731\ldots$ is

$$\sqrt{2} = [1;\ 2,\ 2,\ 2,\ 2,\ 2,\ \ldots]$$

since

$$\sqrt{2} = 1 + \frac{1}{\sqrt{2}+1}\cdot$$

- The continued fraction expansion of $e = 2,718281828459\ldots$ is (Euler)

$$e = [2;\ 1,\ 2,\ 1,\ 1,\ 4,\ 1,\ 1,\ 6,\ 1,\ 1,\ 8,\ 1,\ 1,\ 10,\ 1\ldots]$$

- The development of $\pi = 3,1415926535898\ldots$ starts with

$$\pi = [3;\ 7,\ 15,\ 1,\ 292,\ 1,\ 1,\ 1,\ 2,\ 1,\ 3,\ 1,\ 14,\ 2,\ 1,\ 1\ldots]$$

Rational approximations to $\log_2 3$

- The logarithm in basis 2 of 3 :

$$\log_2 3 = (\log 3)/\log 2 = 1,58496250072\dots$$

is the solution x of the equation $2^x = 3$

- Rational approximations a/b to $\log_2 3$ correspond to powers of 2 which are close to powers of 3 :

$$\log_2 3 \simeq a/b, \qquad 2^a \simeq 3^b.$$

- The continued fraction expansion

$$\log_2 3 = [1; 1, 1, 2, 2, 3, 1, 5, \ \dots]$$

produces numerical values which play an important role in musical scales.

Musical scales and Diophantine approximation

- An approximation to $\log_2 3 = 1.5849625\dots$ is :

$$[1; 1, 1, 2, 2] = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2}}}} = \frac{19}{12} = 1,5833\dots$$

- $2^{19} = 524\,288 \simeq 3^{12} = 531\,441$,
  $(3/2)^{12} = 129.74\dots$ is close to $2^7 = 128$.
  Twelve fifths is a bit more than seven octaves
- Comma of Pythagoras : $3^{12}/2^{19} = 1,01364$

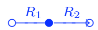
$$1.36\%$$

Further remarkable approximations

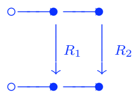
- $5^3 = 125 \simeq 2^7 = 128 \qquad (5/4)^3 = 1,953\ldots \simeq 2$
  Three thirds (ratio $5/4$) produce almost one octave.
- $2^{10} = 1024 \simeq 10^3$.
  • Computers (kilo octets)
  • Acoustic : multiplying the intensity of a sound by 10 means adding 10 decibels.
  Multiplying the intensity by k, amounts to add d decibels with $10^d = k^{10}$.
  Since $2^{10} \simeq 10^3$, doubling the intensity, is close to adding 3 decibels.

Electric networks

- The resistance of a network in series



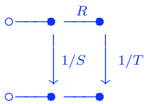
  is the sum $R_1 + R_2$.
- The resistance R of the parallel network



  satisfies
$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

Networks and continued fractions

The resistance U of the circuit



is given by

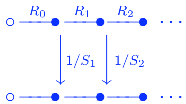
$$U = \cfrac{1}{S + \cfrac{1}{R + \cfrac{1}{T}}}$$

Electric networks, continued fractions and decomposition of a square into squares

- The resistance of the following network is given by a continued fraction

$$[R_0; S_1, R_1, S_2, R_2 \dots]$$

for the circuit



R_i : resistances i series

$1/S_j$: resistances in parallele

- For instance when $R_i = S_j = 1$ we get the quotients of consecutive Fibonacci numbers.
- Electric networks and continued fractions were use to find the first solution to the problem of decomposing a geometric integer square into distinct integer squares.

Number Theory in Science and communication

M.R. Schroeder.
Number theory in science and communication :
with applications in cryptography, physics, digital information, computing and self similarity
Springer series in information sciences **7** 1986.
4th ed. (2006) 367 p.

Quadratic numbers

- The continued fraction expansion of a real number is ultimately periodic if and only if the number is a quadratic number, that means root of a degree 2 polynomial with rational coefficients.
- A real number of the form $\sqrt{d}$ with square-free $d > 0$ has a continued fraction expansion of the form

$$[a_0;\ a_1,\ a_2,\ \dots,\ a_k,\ a_1,\ a_2,\ \dots,\ a_k,\ a_1,\ a_2,\ \dots,]$$

which we write for simplicity

$$[a_0;\ \overline{a_1,\ a_2,\ \dots,\ a_k}].$$

- Hence $\sqrt{2} = [1;\ \overline{2}]$ and $\Phi = \frac{1+\sqrt{5}}{2} = [1;\ \overline{1}]$.

Connexion with the equation $x^2 - dy^2 = \pm 1$

Let d be a square-free positive integer. Consider the Diophantine equation

$$x^2 - dy^2 = \pm 1 \qquad (1)$$

where the unknowns x, y are in $\mathbf{Z}$.

- If (x, y) is a solution, then $(x - \sqrt{d}y)(x + \sqrt{d}y) = 1$, hence x/y is a rational approximation of $\sqrt{d}$ and this approximation is sharper when x is larger.
- This is why a strategy for solving Pell's equation (1) is based on the continued fraction expansion of $\sqrt{d}$.

The simplest example of Pell's equation

A solution to Pell's equation :

$$x^2 - 2y^2 = 1$$

is $x = 3$, $y = 2$, corresponding to the expansion

$$[1; 2] = 1 + \frac{1}{2} = \frac{3}{2}.$$

The number $3 + 2\sqrt{2}$ is a unit of norm 1 in the quadratic number field $\mathbf{Q}(\sqrt{2})$.
The fundamental unit $1 + \sqrt{2}$ has norm -1, it corresponds to the fundamental solution $x = 1$, $y = 1$, of the equation $x^2 - 2y^2 = -1$.

Problem of Brahmagupta (628)

- Brahmasphutasiddhanta : Solve in integers the equation

$$x^2 - 92y^2 = 1$$

- If (x, y) is a solution, then $(x - \sqrt{92}y)(x + \sqrt{92}y) = 1$, hence x/y is a good approximation of $\sqrt{92} = 9,591663046625\ldots$.
- The continued fraction expansion of $\sqrt{92}$ is

$$\sqrt{92} = [9; \overline{1,\ 1,\ 2,\ 4,\ 2,\ 1,\ 1,\ 18}]$$

reference : http ://wims.unice.fr/wims/

- According to the theory, a solution is obtained from

$$[9;\ 1,\ 1,\ 2,\ 4,\ 2,\ 1,\ 1] = \frac{1151}{120}.$$

- Indeed $1151^2 - 92 \cdot 120^2 = 1\,324\,801 - 1\,324\,800 = 1$.

Bhaskara II (12th Century)

- *Lilavati*
- (*Bijaganita*, 1150) $x^2 - 61y^2 = 1$
- $x = 1\,766\,319\,049$, $y = 226\,153\,980$.
  Cyclic method (Chakravala) of Brahmagupta.
- $\sqrt{61} = [7; \overline{1,\ 4,\ 3,\ 1,\ 2,\ 2,\ 1,\ 3,\ 4,\ 1,\ 14}]$
  $[7; 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14, 1, 4, 3, 1, 2, 2, 1, 3, 5] = \frac{1\,766\,319\,049}{226\,153\,980}$
- $[7; 1, 4, 3, 1, 2, 2, 1, 3, 5] = \frac{29\,718}{3\,805}$
- $29\,718^2 = 883\,159\,524, \qquad 61 \cdot 3805^2 = 883\,159\,525$
  solution of $x^2 - 61y^2 = -1$.

Narayana (14th Century)

- Narayana cows (Tom Johnson)
- $x^2 - 103y^2 = 1$
  $x = 227\,528$, $y = 22\,419$.
  $227\,528^2 - 103 \cdot 22\,419^2 = 51\,768\,990\,784 - 51\,768\,990\,783 = 1.$
- $\sqrt{103} = [10; \overline{6, 1, 2, 1, 1, 9, 1, 1, 2, 1, 6, 20}]$
- $[10; 6, 1, 2, 1, 1, 9, 1, 1, 2, 1, 6] = \dfrac{227\,528}{22\,419}$

A reference on the History of Numbers

André Weil
Number theory. :
An approach through history.
From Hammurapi to
Legendre.
Birkhäuser Boston, Inc.,
Boston, Mass., (1984) 375 pp.
MR 85c :01004

Riemannian varieties with negative curvature

- The study of the so-called Pell-Fermat Diophantine equation yield the construction of Riemannian varieties with negative curvature : *arithmetic varieties.*
- Nicolas Bergeron (Paris VI) : "Sur la topologie de certains espaces provenant de constructions arithmétiques"

Further connexions between Diophantine approximation and Diophantine equations

- M. Bennett (1997) :
  For any $p/q \in \mathbf{Q}$,
  $$\left|\sqrt[3]{2} - \frac{p}{q}\right| \geq \frac{1}{4\ q^{2,5}}.$$
- *For any* $(x, y) \in \mathbf{Z}^2$ *with* $x > 0$,
  $$|x^3 - 2y^3| \geq \sqrt{x}.$$
- Previous results : Thue, Siegel, Roth, Baker, Chudnovskii, Easton, Rickert, Voutier,...

Algebraic and transcendental numbers

- A complex number is **algebraic** if it is a root of a non-zero polynomial with rational coefficients

$$a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_n\alpha^n = 0$$

with $a_0, a_1, \ldots, a_n$ rational numbers, not all of which are 0.

- `Examples`
Rational numbers are algebraic : $p/q \in \mathbf{Q}$ is root of $qX - p$.
The number $\sqrt{2}$ is algebraic, root of $X^2 - 2$, and it is not rational.
Also i, root of $X^2 + 1$, is algebraic irrational.
Any root of unity is algebraic, root of some polynomial $X^n - 1$.
- A number is **transcendental** if it is not algebraic.

Ramanujan's approximation for π

- $$\frac{63}{25}\left(\frac{17+15\sqrt{5}}{7+15\sqrt{5}}\right) = 3.141\,592\,653 \quad 805\ldots$$

is a root of $P(x) = 168\,125\,x^2 - 792\,225x + 829\,521$.
- The number

$$\pi = 3.141\,592\,653 \;\; 589\ldots$$

is not root of a polynomial with integer coefficients.

Rational approximation to algebraic numbers

Theorem (Liouville, 1844).

Let α be a real algebraic number. There exists $\kappa > 0$ such that, for any rational number p/q distinct from α with $q \geq 2$,

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{1}{q^\kappa}\cdot$$

Corollary.

Let ξ be a real number. Assume that for any $\kappa > 0$ there exists a rational number p/q with $q \geq 2$ such that

$$0 < \left|\xi - \frac{p}{q}\right| < \frac{1}{q^\kappa}\cdot$$

Then ξ is transcendental.

Liouville numbers

- Definition : A *Liouville number* is a real number such that, for any $\kappa > 0$, there exists a rational number p/q with $q \geq 2$ satisfying

$$0 < \left|\xi - \frac{p}{q}\right| < \frac{1}{q^\kappa}\cdot$$

- In *dynamical systems theory*, a *Diophantine number* (or *a number satisfying a Diophantine condition*) is a real number which is not Liouville : there exists $\kappa \geq 2$ and $C > 0$ such that

$$\left|\xi - \frac{p}{q}\right| \geq \frac{C}{q^\kappa}\cdot$$

- J.C. Yoccoz : *Conjugaison différentiable des difféomorphismes du cercle dont le nombre de rotation vérifie une condition diophantienne.* Ann. scient. Éc. Norm. Sup. $4^{\text{è}}$ série, t. **17** (1984), 333-359.

Examples of transcendental numbers

- The number
$$\sum_{n\geq 0} 2^{-n!}$$
is transcendental (Liouville, 1844)
- The number e is transcendental (Hermite, 1873)
- The number π is transcendental (Lindemann, 1882)
Consequence : *negative solution of the problem of squaring the circle.*
- **Hermite–Lindemann Theorem**. For algebraic α and β with $\alpha \neq 0$, $\alpha \neq 1$, $\beta \neq 0$, the numbers $\log \alpha$ and e^β are transcendental.
- Further examples : $\log 2$, $e^{\sqrt{2}}$ are transcendental.

Hilbert's seventh problem

- Seventh of Hilbert's 23 problems (1900) : *Prove the transcendence of* $\log \alpha_1 / \log \alpha_2$ *and of* α^β *for algebraic* α *and* β.
- Examples : $2^{\sqrt{2}}$ and $e^\pi = (-1)^{-i}$.
Recall $\alpha^\beta = \exp(\beta \log \alpha)$.
- Solution by A.O. Gel'fond and Th. Schneider in 1934.

Conseuquences of the Gel'fond–Schneider Theorem

Consequences : *transcendence of*

$$2^{\sqrt{2}}, \quad e^{\pi} = (e^{i\pi})^{-i}, \quad \frac{\log 2}{\log 3}$$

and of the Ramanujan number $e^{\pi\sqrt{163}} = a - 10^{-12}b$ *where*

$$a = 262\ 537\ 412\ 640\ 768\ 744 \in \mathbf{Z}, \quad b = 0.7499274\cdots$$

Connexion with Diophantine approximation

- A.O. Gel'fond : lower bounds for $|\alpha_1^{\beta} - \alpha_2|$.
- Special case $\beta \in \mathbf{Q}$: lower bounds for

$$|\alpha_1^{-b_1/b_2} - \alpha_2|$$

- If $\alpha_1^{-b_1/b_2} \simeq \alpha_2$ then $\alpha_1^{b_1} \simeq \alpha_2^{-b_2}$ and $\alpha_1^{b_1}\alpha_2^{b_2} \simeq 1$.
- Lower bounds for $|2^p - 3^q|$ and for

$$\left|\log_2 3 - \frac{p}{q}\right|.$$

Gelfond-Baker's method

- *Gel'fond–Baker method :* lower bounds for

$$|e^{\beta_0}\alpha_1^{\beta_1}\cdots\alpha_n^{\beta_n}-1|.$$

- *Special case :*

$$|e^{\beta}-\alpha|$$

with algebraic α and β
- *In particular when α and β are rational numbers, in particular when they are rational integers :*

$$|e^{b}-a|$$

Mahler's problem (1967)

- For a and b positive integers,

$$|e^{b}-a|>a^{-c}?$$

- Stronger conjecture :

$$|e^{b}-a|>b^{-c}?$$

- K. Mahler (1953, 1967), M. Mignotte (1974), F. Wielonsky (1997) :

$$|e^{b}-a|>b^{-20b}$$

- Joint work with Yu.V. Nesterenko (1996) for a and b rational numbers, refinement by S. Khemira (2005).

Exact rounding of the elementary functions

- Define $H(p/q) = \max\{|p|,\ q)$. Then for a and b in $\mathbf{Q}$ with $b \neq 0$,

$$|e^b - a| \geq \exp\{-1, 3 \cdot 10^5 (\log A)(\log B)\}$$

where $A = \max\{H(a),\ A_0\}$, $B = \max\{H(b),\ 2\}$.

- Applications in theoretical computer science :
Muller, J-M. ; Tisserand, A. –
Towards exact rounding of the elementary functions.
Alefeld, Goetz (ed.) et al.,
Scientific computing and validated numerics.
Proceedings of the international symposium on scientific computing, computer arithmetic and validated numerics SCAN-95, Wuppertal, Germany, September 26-29, 1995.
Berlin : Akademie Verlag. Math. Res. 90, 59-71 (1996).

Applications in theoretical computer science

Computer Arithmetic
—
Arénaire project
http ://www.ens-lyon.fr/LIP/Arenaire/

Validated scientific computing
Arithmetic. reliability, accuracy, and speed
Improvement of the available arithmetic on computers, processors, dedicated or embedded chips
Getting more accurate results or getting them more quickly
Power consumption, reliability of numerical software.

Further applications of Diophantine Approximation

- Hua Loo Keng, Wang Yuan
  Application of number theory to numerical analysis
  Springer Verlag 1981

  Equidistribution modulo 1, discrepancy, numerical integration, interpolation, approximate solutions to integral and differential equations.

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Department of Mathematics
Pondicherry University

An introduction to Diophantine analysis and transcendental number theory

Michel Waldschmidt

Institut de Mathématiques de Jussieu & CIMPA
http ://www.math.jussieu.fr/∼miw/