Society for
Special Functions
& their Applications

International Conference on Special Functions & Applications (ICSFA-2023)

(XXIInd Annual Meeting of the Society for Special Functions & their Applications, India)
&
Symposium on Algebra & Algebraic Combinatorics

2 -4 November, 2023

Organized by
Department of Mathematics, Ramanujan School of Mathematical Sciences
Pondicherry University, Puducherry, India

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Interpolation problems

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ICSFA 2023 — Abstract

Given a sequence $(s_n)_{n\geq 0}$ of complex numbers, a sequence $(t_n)_{n\geq 0}$ of nonnegative integers and a sequence $(E_n)_{n\geq 0}$ of subsets of $\mathbb{C}$, the corresponding interpolation problem is to investigate the set of entire functions f such that $f^{(t_n)}(s_n) \in E_n$ for all n. The existence problem is when each E_n consists of a single element. The unicity problem is for $E_n = \{0\}$ for all n. The arithmetic problem of integer valued functions is for $E_n = \mathbb{Z}$ for all n. Classical examples are those investigated by Abel (1802–1829, published posthumously in 1881), Halphén (1882), Gontcharoff (1930), Lidstone (1932), Poritsky (1932), Whittaker (1934), Schoenberg (1936), Straus (1950), Macintyre (1954), Buck (1955), Sato (1964). Many problems are open; we discuss some of them.

ICSFA 2016

2016, September 9-11, XVth Annual Conference of Society for Special Functions and their Applications (ICSFA-2016), Department of Applied Sciences and Humanities, Faculty of Engineering and Technology, Jamia Millia Islamia (Central University), New Delhi.

September 9, 2016 — R. P. Agarwal Memorial Lecture:
Irrationality and transcendence of values of special functions.

Proceedings, New Delhi, India. Editors: A. K. Agarwal, M. A. Pathan, Subuhi Khan, Ch. Wali Mohd.

http://www.ssfaindia.org/conf_proc.htm

http://www.imj-prg.fr/~michel.waldschmidt/AgendaArchives.html

ICSFA 2018

2018, November 22 - 24: 17th Annual Meeting of SSFA - International Conference on Special Functions & Applications (ICSFA-2018), Amal Jyothi College on Engineering, Kanjirapalli, Kottayam (Kerala).

Lecture on *Linear recurrence sequences: an introduction.*

November 23, 2018: M. A. Pathan, AMU, Aligarh.
On Generalization of Taylor's series, Riemann Zeta Functions and Bernoulli Polynomials

http://www.imj-prg.fr/~michel.waldschmidt/AgendaArchives.html

ICSFA 2019

October 21 - 23, 2019: Bikaner, Rajasthan

International Conference on Special Functions & Applications (ICSFA-2019). Department of Mathematics, University College of Engineering and Technology.

Lecture on *Lidstone series, generalisations and arithmetic applications.*

http://www.imj-prg.fr/~michel.waldschmidt/AgendaArchives.html

ICSFA 2021 **(online)**

Decembre 22-23, 2021, Department of Mathematics, University of Kerala

ICSFA-2021 (International Conference on Special Functions and Applications) and 20th Annual meeting of the Society for Special functions and their Applications.

Lecture on *Lidstone expansion in several variables.*

http://www.imj-prg.fr/~michel.waldschmidt/AgendaArchives.html

ICSFA 2022

November 26 - 28, 2022, Department of Studies in Mathematics, University of Mysore, India.
International Conference on Special Functions & Applications (ICSFA-2022) (XXIst Annual Meeting of the Society for Special Functions & their Applications, India).

Lecture on *A multidimensional analog of Lidstone interpolation theory.* .

http://ssfaindia.org/icsfa2022/

International Conference on Special Functions & Applications (ICSFA-2023).
http://ssfaindia.org.in/icsfa2023/
Department of Mathematics, Ramanujan School of Mathematical Sciences
Pondicherry University, Puducherry, India

Interpolation Problems

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Examples of interpolation problems

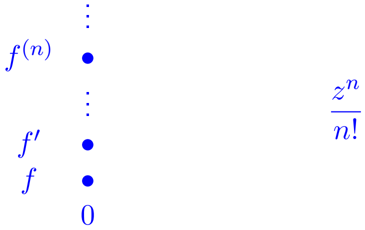
- (Lidstone):
$$f^{(2n)}(0) = a_n, \quad f^{(2n)}(1) = b_n \quad \text{for} \quad n \geq 0.$$

- (Whittaker):
$$f^{(2n+1)}(0) = a_n, \quad f^{(2n)}(1) = b_n \quad \text{for} \quad n \geq 0.$$

- (Poritsky): For $m \geq 2$ and $\sigma_0, \ldots, \sigma_{m-1}$ in $\mathbb{C}$,
$$f^{(mn)}(\sigma_j) = a_{nj} \quad \text{for} \quad n \geq 0 \quad \text{and} \quad j = 0, 1, \ldots, m-1.$$

- (Gontcharoff): For $(\sigma_n)_{n \geq 0}$ a sequence of complex numbers,
$$f^{(n)}(\sigma_n) = a_n \quad \text{for} \quad n \geq 0.$$

Taylor expansion: vertical



Hurwitz functions

Kakeya, S. (1916).
Notes on the maximum modulus of a function.
Tohoku Math. J., 10:68–72.

Pólya, G. (1921).
Über die kleinsten ganzen Funktionen, deren sämtliche Derivierten im Punkte $z = 0$ ganzzahlig sind. (Auszug aus einem an T. Kubota gerichteten Briefe).
Tohoku Math. J., 19:65–68.

Pólya's expansion: horizontal

$$\begin{array}{cccccccc} f & \bullet & \bullet & \bullet & \cdots & \bullet & \cdots \\ & 0 & 1 & 2 & \cdots & m & \cdots \end{array}$$

$$1, z, \frac{z(z-1)}{2}, \ldots, \frac{z(z-1)\cdots(z-m+1)}{m!}, \ldots$$

Integer–valued entire functions. Calculus of finite differences.

Pólya, G. (1915).
Über ganzwertige ganzen Funktionen.
Rend. circ. math. Palermo, 40:1–16.

G.H. Hardy (1917), A. Selberg (1941), C. Pisot (1942).

2–point Hurwitz functions

$\vdots$	$\vdots$	$\vdots$
$f^{(n)}$	$\bullet$	$\bullet$
$\vdots$	$\vdots$	$\vdots$
f'	$\bullet$	$\bullet$
f	$\bullet$	$\bullet$
	0	1

Gel'fond, A. O. (1934).
Über Potenzreihen mit ganzen Koeffizienten.
Wiss. Ber. Moskauer Univ. 2, 29-33 (1934).

2–times integer–valued functions

$$\begin{array}{ccccccc} f' & \bullet & \bullet & \bullet & \cdots & \bullet & \cdots \\ f & \bullet & \bullet & \bullet & \cdots & \bullet & \cdots \\ & 0 & 1 & 2 & \cdots & m & \cdots \end{array}$$

Gel'fond, A. O. (1929).
Sur les propriétés arithmétiques des fonctions entières.
Tohoku Math. J., 30:280–285.

Selberg, A. (1941).
Über einen Satz von A. Gel'fond.
Arch. Math. Naturvid., 44:159–170.

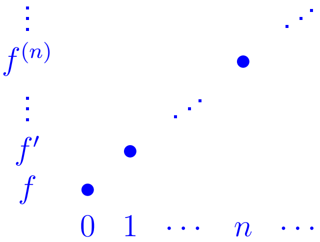
Utterly integer–valued entire functions

$$
\begin{array}{cccccc}
\vdots & \vdots & \vdots & & \vdots & \\
f^{(n)} & \bullet & \bullet & \cdots & \bullet & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \\
f' & \bullet & \bullet & \cdots & \bullet & \cdots \\
f & \bullet & \bullet & \cdots & \bullet & \cdots \\
 & 0 & 1 & \cdots & m & \cdots
\end{array}
$$

Straus, E. G. (1951).
On the polynomials whose derivatives have integral values at the integers.
Proc. Amer. Math. Soc., 2:23–27.

Fridman, G. A. (1968).
Entire integer-valued functions.
Mat. Sb. (N.S.), 75 (117):417–431.

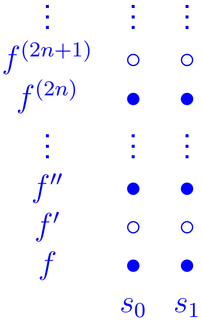
Abel interpolation



Abel, N. H. (1881).
Œuvres complètes de Niels Henrik Abel. Vol. 1 and 2.
Cambridge: Cambridge University Press, reprint of the new edition published 1881 by grondahl and son, 2012 edition.

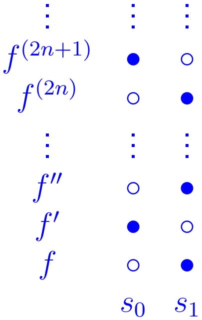
Halphén, G. (1882).
Sur une série d'Abel.
C. R. Acad. Sci., Paris, 93:1003–1005.
Bull. Soc. Math. Fr., 10:67–87.

Lidstone interpolation



Lidstone, G. J. (1930).
Notes on the extension of Aitken's theorem (for polynomial interpolation) to the Everett types.
Proc. Edinb. Math. Soc., II. Ser., 2:16–19.

Whittaker interpolation



Whittaker, J. M. (1933).
On Lidstone's series and two-point expansions of analytic functions.
Proc. Lond. Math. Soc. (2), 36:451–469.

Whittaker, J. M. (1935).
Interpolatory function theory, volume 33.
Cambridge University Press, Cambridge.

Poritsky interpolation (3 points)

$\vdots$	$\vdots$	$\vdots$	$\vdots$
$f^{(3n+2)}$	○	○	○
$f^{(3n+1)}$	○	○	○
$f^{(3n)}$	●	●	●
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$f^{(\text{iv})}$	○	○	○
f'''	●	●	●
f''	○	○	○
f'	○	○	○
f	●	●	●
	s_0	s_1	s_2

Poritsky, H. (1932).
On certain polynomial and other approximations to analytic functions.
Trans. Amer. Math. Soc., 34(2):274–331.

Gontcharoff interpolation (3 points)

$\vdots$	$\vdots$	$\vdots$	$\vdots$
$f^{(3n+2)}$	○	○	●
$f^{(3n+1)}$	○	●	○
$f^{(3n)}$	●	○	○
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$f^{(\text{iv})}$	○	●	○
f'''	●	○	○
f''	○	○	●
f'	○	●	○
f	●	○	○
	s_0	s_1	s_2

Gontcharoff, W. (1930).
Recherches sur les dérivées successives des fonctions analytiques. Généralisation de la série d'Abel.
Ann. Sci. Éc. Norm. Supér. (3), 47:1–78.

Abel–Gontcharoff interpolation: $f^{(n)}(w_n)$

We start with any sequence $\mathbf{w} = (w_n)_{n\geq 0}$ of complex numbers. We define a sequence of polynomials $(\Omega_{w_0,w_1,\dots,w_{n-1}})_{n\geq 0}$ in $\mathbb{C}[z]$ so that any polynomial P can be written as a finite sum

$$P(z) = \sum_{n\geq 0} P^{(n)}(w_n)\Omega_{n;\mathbf{w}}(z).$$

The condition is

$$\Omega^{(k)}_{n;\mathbf{w}}(w_k) = \delta_{kn}$$

for $n \geq 0$ and $k \geq 0$.

Alternatively, one may proceed as follows: we set $\Omega_\emptyset = 1$, $\Omega_{w_0}(z) = z - w_0$, and, for $n \geq 1$, we define $\Omega_{w_0,w_1,w_2,\dots,w_n}(z)$ as the polynomial of degree $n+1$ which is the primitive of $\Omega_{w_1,w_2,\dots,w_n}$ vanishing at w_0.

The sequence of polynomials $\Omega_{n;\mathbf{w}}$

For $n \geq 0$, we write $\Omega_{n;\mathbf{w}}$ for $\Omega_{w_0,w_1,\ldots,w_{n-1}}$, a polynomial of degree n which depends only on the first n terms of the sequence $\mathbf{w}$. The leading term of $\Omega_{n;\mathbf{w}}$ is $(1/n!)z^n$.
In particular, for $N \geq 0$ we have

$$\frac{z^N}{N!} = \sum_{n=0}^{N} \frac{1}{(N-n)!} w_n^{N-n} \Omega_{n;\mathbf{w}}(z).$$

This gives an inductive formula defining $\Omega_{N;\mathbf{w}}$: for $N \geq 0$,

$$\Omega_{N;\mathbf{w}}(z) = \frac{z^N}{N!} - \sum_{n=0}^{N-1} \frac{1}{(N-n)!} w_n^{N-n} \Omega_{n;\mathbf{w}}(z).$$

Iterated integrals

We also have

$$\Omega_{w_0,w_1,\dots,w_n}(z) = \Omega_{0,w_1-w_0,w_2-w_0,\dots,w_n-w_0}(z-w_0).$$

With $w_0 = 0$, the first polynomials are given by

$$\begin{aligned}
2!\Omega_{0,w_1}(z) &= (z-w_1)^2 - {w_1}^2, \\
3!\Omega_{0,w_1,w_2}(z) &= (z-w_2)^3 - 3(w_1-w_2)^2 z + {w_2}^3, \\
4!\Omega_{0,w_1,w_2,w_3}(z) &= (z-w_3)^4 - 6(w_2-w_3)^2(z-w_1)^2 \\
&\quad - 4(w_1-w_3)^3 z + 6{w_1}^2(w_2-w_3)^2 - {w_3}^4.
\end{aligned}$$

From the definition we deduce the following formula, involving iterated integrals

$$\Omega_{w_0,w_1,\dots,w_{n-1}}(z) = \int_{w_0}^{z} \mathrm{d}t_1 \int_{w_1}^{t_1} \mathrm{d}t_2 \cdots \int_{w_{n-1}}^{t_{n-1}} \mathrm{d}t_n.$$

A determinant

Gontcharoff:

$$\Omega_{w_0,w_1,\ldots,w_{n-1}}(z) = (-1)^n \begin{vmatrix} 1 & \frac{z}{1!} & \frac{z^2}{2!} & \cdots & \frac{z^{n-1}}{(n-1)!} & \frac{z^n}{n!} \\ 1 & \frac{w_0}{1!} & \frac{{w_0}^2}{2!} & \cdots & \frac{{w_0}^{n-1}}{(n-1)!} & \frac{{w_0}^n}{n!} \\ 0 & 1 & \frac{w_1}{1!} & \cdots & \frac{{w_1}^{n-2}}{(n-2)!} & \frac{{w_1}^{n-1}}{(n-1)!} \\ 0 & 0 & 1 & \cdots & \frac{{w_2}^{n-3}}{(n-3)!} & \frac{{w_2}^{n-2}}{(n-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \frac{w_{n-1}}{1!} \end{vmatrix}.$$

Examples

With the sequence $\mathbf{w} = (1, 0, 1, 0, \dots, 0, 1, \dots)$, we recover the Whittaker polynomials associated with the interpolation problem

$$f^{(2n+1)}(0) = a_n, \quad f^{(2n)}(1) = b_n \quad \text{for} \quad n \geq 0.$$

Another example, considered by N. Abel is the arithmetic progression $\mathbf{w} = (a + nt)_{n \geq 0}$ with a in $\mathbb{C}$ and t in $\mathbb{C} \setminus \{0\}$, where

$$\Omega_{n;\mathbf{w}}(z) = \frac{1}{n!}(z - a)(z - a - nt)^{n-1}$$

for $n \geq 1$, which satisfies

$$\Omega'_{n;\mathbf{w}}(z) = \Omega_{n-1;\mathbf{w}}(z - t).$$

Standard pairs of polynomials (Whittaker)

Let $\mathcal{S} \subset \{0,1\} \times \mathbb{N}$. Let $\underline{p} = (p_1, p_2, \ldots)$ be the sequence of integers such that $(1, p_i) \in \mathcal{E}$ and let $\underline{q} = (q_1, q_2, \ldots)$ be the sequence of integers such that $(0, q_i) \in \mathcal{E}$.
According to Whittaker, a set of polynomials

$$(\underline{\pi}, \underline{\zeta}) = \{\pi_1, \pi_2, \ldots, \pi_m, \ldots \; ; \zeta_1, \zeta_2, \ldots, \zeta_m, \ldots\}$$

is a *standard set of polynomials for* $\mathcal{S}$ if any polynomial $f \in \mathbb{C}[z]$ can be written (in a unique way)

$$f(z) = \sum_{n \geq 0} \left(f^{(p_n)}(1)\pi_n(z) + f^{(q_n)}(0)\zeta_n(z) \right)$$

Equivalent definition

Given two sequences $\underline{p} = (p_n)_{n\geq 0}$ and $\underline{q} = (q_n)_{n\geq 0}$ of nonnegative integers, a *standard set of polynomials* for $(\underline{p}, \underline{q})$ is a pair $(\underline{\pi}, \underline{\zeta})$ given by two sequences $\underline{\pi} = (\pi_n)_{n\geq 0}$ and $\underline{\zeta} = (\zeta_n)_{n\geq 0}$ of polynomials such that, for $n, k \geq 0$,

$$\pi_n^{(p_k)}(1) = \delta_{nk}, \quad \pi_n^{(q_k)}(0) = 0$$

and

$$\zeta_n^{(p_k)}(1) = 0, \quad \zeta_n^{(q_k)}(0) = \delta_{nk}$$

Whittaker classification

Here is Whittaker classification of *complete, indeterminate* and *redundant* sequences, involving standard sets of polynomials.

If there exists a unique standard set of polynomials $(\underline{\pi}, \underline{\zeta})$, then $(\underline{p}, \underline{q})$ is called *complete*, and any polynomial f can be written in a unique way as a finite sum

$$f(z) = \sum_{n \geq 0} f^{(p_n)}(1)\pi_n(z) + \sum_{n \geq 0} f^{(q_n)}(0)\zeta_n(z).$$

If there are several solutions $(\underline{\pi}, \underline{\zeta})$, then $(\underline{p}, \underline{q})$ is called *indeterminate*.

If there is no solution $(\underline{\pi}, \underline{\zeta})$, then $(\underline{p}, \underline{q})$ is called *redundant*.

Complete sets

For $m \geq 1$, define

$$P(m) = \#\{0 \leq n \leq m-1\ ;\ (0,n) \in \mathcal{E}\}$$

and

$$Q(m) = \#\{0 \leq n \leq m-1\ ;\ (1,n) \in \mathcal{E}\}.$$

Then the set $\mathcal{E}$ is complete if and only if

$$P(m) + Q(m) \geq m$$

for all $m \geq 1$, and there are infinitely many m with $P(m) + Q(m) = m$.

Indeterminate sets

If the set $\mathcal{E}$ is indeterminate, then $P(m) + Q(m) < m$ for infinitely many m.

Some research problems

- In the complete case, study the unique standard set of polynomials $(\underline{\pi}, \underline{\zeta})$.

- Extend to more than two points.

- Generalize to several variables.

- Investigate the arithmetic question of integer valued entire functions in these situations.

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