Michel Waldschmidt Interpolation, January 2021

Exercices: hints, solutions, comments

Second course

1. Prove the two lemmas on entire functions p. 16.

$\boxed{\textbf{1.}}$

Lemma. *An entire function f is periodic of period $\omega \neq 0$ if and only if there exists a function g analytic in $\mathbb{C}^\times$ such that $f(z) = g(\mathrm{e}^{2i\pi z/\omega})$.*

`Solution.` The map $z \mapsto \mathrm{e}^{i\pi z}$ is analytic and surjective. The condition $\mathrm{e}^{i\pi z} = \mathrm{e}^{i\pi z'}$ implies $f(z) = f(z')$. Hence there exists a unique map $g : \mathbb{C}^\times \to \mathbb{C}$ such that $g(\mathrm{e}^{2i\pi z}) = f(z)$.

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, "f"] \arrow[d, "\mathrm{e}^{2i\pi z}"] & \mathbb{C} \\
\mathbb{C}^\times \arrow[ur, dotted, "g"'] &
\end{tikzcd}
$$

Let $t \in \mathbb{C}^\times$ and let $z \in \mathbb{C}$ be such that $t = \mathrm{e}^{2i\pi z}$. Then $g(t) = f(z)$ and $g'(t) = \frac{1}{2\pi} f'(z)$.

This proves the first lemma.

Lemma. *If g is an analytic function in $\mathbb{C}^\times$ and if the entire function $g(\mathrm{e}^{2i\pi z/\omega})$ has a type $< 2(N+1)\pi/|\omega|$, then $t^N g(t)$ is a polynomial of degree $\leq 2N$.*

Therefore, if $g(\mathrm{e}^{2i\pi z/\omega})$ has a type $< 2\pi/|\omega|$, then g is constant.

`Solution.` Assume that the function $f(z) = g(\mathrm{e}^{2i\pi z/\omega})$ has a type $\tau < 2(N+1)\pi/|\omega|$. Let $t \in \mathbb{C}^\times$. Write $t = |t|\mathrm{e}^{i\theta}$ with $|\theta| \leq \pi$. Set

$$z = \frac{\omega}{2i\pi}(\log|t| + i\theta),$$

so that $t = \mathrm{e}^{2i\pi z/\omega}$. For any $\epsilon_1 > 0$, we have

$$|z| \leq \left(\frac{\omega}{2\pi} + \epsilon_1\right) |\log|t||$$

for sufficiently large $|t|$ and also for sufficiently small $|t|$. We deduce

$$\log|g(t)| = \log|f(z)| \leq (\tau + \epsilon_2)|z| \leq \left(\frac{\omega\tau}{2\pi} + \epsilon_3\right) |\log|t||.$$

Notice that

$$\frac{\omega\tau}{2\pi} < N + 1.$$

Hence $|g|_r \leq e^{\alpha r}$ for sufficiently large r and also for sufficiently small $r > 0$ with $\alpha < N + 1$. Write

$$g(t) = \sum_{n \in \mathbb{Z}} b_n t^n.$$

From

$$b_n = \frac{1}{2\pi} \int_{|t|=r} g(t) \frac{\mathrm{d}t}{t^{n+1}}$$

we deduce Cauchy's inequalities

$$|b_n|r^n \le \frac{1}{2\pi}|g|_r.$$

For $n > N$, we use these inequalities with $r \to \infty$ while for $n < -N$, we use these inequalities with $r \to 0$. We deduce $b_n = 0$ for $|n| \ge N+1$. Hence

$$g(t) = \frac{1}{t^N}A(t) + B(t)$$

where A and B are polynomials of degree $\le N$.

2. Check $c_n'' = c_{n-1}$ for $n \ge 1$ p. 22.

$\boxed{\textbf{2.}}$ The function

$$F(z,t) = \frac{\mathrm{e}^{tz} - \mathrm{e}^{-tz}}{\mathrm{e}^{t} - \mathrm{e}^{-t}} = \sum_{n\ge 0} c_n(z)t^{2n}$$

satisfies

$$\left(\frac{\partial}{\partial z}\right)^2 F(z,t) = t^2F(z,t).$$

Since

$$\left(\frac{\partial}{\partial z}\right)^2 F(z,t) = \sum_{n\ge 0} c_n''(z)t^{2n} = c_0''(z) + c_1''(z)t^2 + c_2''(z)t^4 + \cdots$$

and

$$t^2F(z,t) = \sum_{n\ge 0} c_n(z)t^{2n} = c_0(z)t^2 + c_1(z)t^4 + c_2(z)t^6 + \cdots$$

we deduce $c_0''(z) = 0$ and $c_n''(z) = c_{n-1}(z)$ for $n \ge 1$.
As a matter of fact, $c_0(z) = \Lambda_0(z) = z$, $c_n(z) = \Lambda_n(z)$ for $n \ge 0$.

3. Let S be a positive integer and let $z \in \mathbb{C}$. Using Cauchy's residue Theorem, compute the integral (see p. 26)

$$\frac{1}{2\pi i}\int_{|t|=(2S+1)\pi/2} t^{-2n-1}\frac{\mathrm{sh}(tz)}{\mathrm{sh}(t)}\mathrm{d}t.$$

$\boxed{\textbf{3.}}$ The poles of the function

$$t \mapsto \frac{\mathrm{sh}(tz)}{\mathrm{sh}(t)} = \frac{\mathrm{e}^{tz} - \mathrm{e}^{-tz}}{\mathrm{e}^{t} - \mathrm{e}^{-t}}$$

are the complex numbers t such that $\mathrm{e}^{2t} = 1$, namely $t \in i\pi\mathbb{Z}$.

The poles inside $|t| \le (2S+1)\pi/2$ are the $i\pi s$ with $-S \le s \le S$.

The residue at $t = 0$ of $t^{-2n-1}\frac{\mathrm{sh}(tz)}{\mathrm{sh}(t)}$ is the coefficient ot t^{-2n} in the Taylor expansion of $\frac{\mathrm{sh}(tz)}{\mathrm{sh}(t)}$, hence it is $\Lambda_n(z)$.

Let s be an integer in the range $1 \le s \le S$. Write $t = i\pi s + \epsilon$. Then

$$\mathrm{e}^t = (-1)^s(1+\epsilon+\cdots), \quad \mathrm{e}^{-t} = (-1)^s(1-\epsilon+\cdots,) \quad \mathrm{e}^t - \mathrm{e}^{-t} = (-1)^s 2\epsilon + \cdots,$$

and

$$\mathrm{e}^{tz} = \mathrm{e}^{i\pi sz}, \quad \mathrm{e}^{-tz} = \mathrm{e}^{-i\pi sz},$$

so that

$$\frac{\mathrm{e}^{tz}-\mathrm{e}^{-tz}}{\mathrm{e}^{t}-\mathrm{e}^{-t}} = (-1)^s \frac{i\sin(\pi s)}{\epsilon} + \cdots$$

Therefore the residue $t = i\pi s$ of $t^{-2n-1}\frac{\mathrm{sh}(tz)}{\mathrm{sh}(t)}$ is

$$(-1)^{n+s}(\pi s)^{-2n-1}.$$

For $-S \le s \le -1$, the residue at $i\pi s$ is the same.

This proves the formula p. 26 :

$$\Lambda_n(z) = (-1)^n \frac{2}{\pi^{2n+1}} \sum_{s=1}^{S} \frac{(-1)^s}{s^{2n+1}} \sin\big(s\pi z\big) + \frac{1}{2\pi i}\int_{|t|=(2S+1)\pi/2} t^{-2n-1}\frac{\mathrm{sh}(tz)}{\mathrm{sh}(t)}\mathrm{d}t$$

for $S = 1, 2, \ldots$ and $z \in \mathbb{C}$.

4. Prove the proposition p. 31 :
Let f be an entire function. The two following conditions are equivalent.
(i) $f^{(2k)}(0) = f^{(2k)}(1) = 0$ *for all* $k \ge 0$.
(ii) f *is the sum of a series*

$$\sum_{n\ge 1} a_n \sin(n\pi z)$$

which converges normally on any compact.
Prove also the following result :
Let f be an entire function. The two following conditions are equivalent.
(i) $f^{(2k+1)}(0) = f^{(2k)}(1) = 0$ *for all* $k \ge 0$.
(ii) f *is the sum of a series*

$$\sum_{n\ge 1} a_n \cos\left(\frac{(2n+1)\pi}{2} z\right)$$

which converges normally on any compact.

$\boxed{\mathbf{4.}}$
(a) For $n \ge 1$, the function $z \mapsto \sin(n\pi z)$ satisfies (i). Hence (ii) implies (i).

Let us check that (i) implies (ii). The conditions $f^{(2k)}(0) = 0$ for all $k \ge 0$ mean $f(-z) = -f(z)$. The conditions $f^{(2k)}(1) = 0$ for all $k \ge 0$ mean $f(1+z) = -f(1-z)$. Hence $f^{(2k)}(0) = f^{(2k)}(1) = 0$ for all $k \ge 0$ imply $f(z+2) = f(z)$, which means that f is periodic of period 2. Since f is an entire function, from the first lemma p. 16, there exists a function g analytic in $\mathbb{C}^\times$ such that $f(z) = g(\mathrm{e}^{i\pi z})$. Now the condition $f(z) = -f(-z)$ implies $g(1/t) = -g(t)$. Let us write

$$g(t) = \sum_{n\in\mathbb{Z}} b_n t^n.$$

The Laurent series on the right hand side converges normally on every compact in $\mathbb{C}^\times$. The condition $g(1/t) = -g(t)$ implies $b_{-n} = -b_n$ for all $n \in \mathbb{Z}$, hence $b_0 = 0$ and

$$g(t) = \sum_{n\ge 1} b_n\left(t^n - t^{-n}\right)$$

which implies condition (ii) with $a_n = 2ib_n$.
(b) For $n \ge 1$, the function $z \mapsto \cos\left(\frac{(2n+1)\pi}{2} z\right)$ satisfies (i). Hence (ii) implies (i).

Let us check that (i) implies (ii). The conditions $f^{(2k+1)}(0) = 0$ for all $k \ge 0$ mean $f(-z) = f(z)$. The conditions $f^{(2k)}(1) = 0$ for all $k \ge 0$ mean $f(1+z) = f(1-z)$. We deduce that f

is periodic of period 4. Since f is an entire function, from the first lemma p. 16, there exists a function g analytic in $\mathbb{C}^\times$ such that $f(z) = g(\mathrm{e}^{i\pi z/2})$. Now the condition $f(z) = f(-z)$ implies $g(1/t) = g(t)$. We deduce in the same way as above

$$g(t) = \sum_{n\geq 1} b_n \left(t^n + t^{-n}\right)$$

which implies condition (ii).

5. Complete the three proofs of the Lemma p. 33.

$\boxed{5.}$ **Lemma.** *Let f be a polynomial satisfying*

$$f^{(2n+1)}(0) = f^{(2n)}(1) = 0 \text{ for all } n \geq 0.$$

Then $f = 0$.

Let f be a polynomial satisfying

$$f^{(2n+1)}(0) = f^{(2n)}(1) = 0 \text{ for all } n \geq 0.$$

• *First proof* By induction on the degree of the polynomial f.

If f has degree ≤ 1, say $f(z) = a_0 z + a_1$, the conditions $f'(0) = f(1) = 0$ imply $a_0 = a_1 = 0$, hence $f = 0$.

If f has degree $\leq n$ with $n \geq 2$ and satisfies the hypotheses, then f'' also satisfies the hypotheses and has degree $< n$, hence by induction $f'' = 0$ and therefore f has degree ≤ 1. The result follows.

• *Second proof* The assumption $f^{(2n+1)}(0) = 0$ for all $n \geq 0$ means that f is an even function : $f(-z) = f(z)$. The assumption $f^{(2n)}(1) = 0$ for all $n \geq 0$ means that $f(1-z)$ is an odd function : $f(1-z) = -f(1+z)$. We deduce $f(z+2) = f(1+z+1) = -f(1-z-1) = -f(-z) = -f(z)$, hence $f(z+4) = f(z)$; it follows that the polynomial f is periodic, and therefore it is a constant. Since $f(1) = 0$, we conclude $f = 0$.

• *Third proof* Write

$$f(z) = a_0 + a_2 z^2 + a_4 z^4 + a_6 z^6 + a_8 z^6 + \cdots + a_{2n} z^{2n} + \cdots$$

(finite sum). We have $f(1) = f''(1) = f^{(\mathrm{iv})}(1) = \cdots = 0$:

$$\begin{array}{ccccccc}
a_0 & +a_2 & +a_4 & +a_6 & +\cdots & +a_{2n} & +\cdots = 0 \\
 & 2a_2 & +12a_4 & +30a_5 & +\cdots & +2n(2n-1)a_{2n} & +\cdots = 0 \\
 & & 24a_4 & +360a_6 & +\cdots & +\frac{(2n)!}{(2n-4)!}a_{2n} & +\cdots = 0 \\
 & & & & & \ddots & \vdots
\end{array}$$

The matrix of this system is triangular with maximal rank.

6. Let $(M_n(z))_{n\geq 0}$ and $(\widetilde{M}_n(z))_{n\geq 0}$ be two sequences of polynomials such that any polynomial $f \in \mathbb{C}[z]$ has a finite expansion

$$f(z) = \sum_{n=0}^{\infty} \left(f^{(2n)}(1) M_n(z) + f^{(2n+1)}(0) \widetilde{M}_n(z) \right),$$

with only finitely many nonzero terms in the series (see p. 34). Check

$$\widetilde{M}_n(z) = -M'_{n+1}(1-z)$$

for $n \geq 0$.
`Hint:` Consider $f'(1-z)$.

$\boxed{\mathbf{6.}}$ Define $\widetilde{f}(z) = f'(1-z)$. Write

$$f(z) = \sum_{n=0}^{\infty} \left(f^{(2n)}(1) M_n(z) + f^{(2n+1)}(0) \widetilde{M}_n(z) \right).$$

Then

$$f'(z) = \sum_{n=0}^{\infty} \left(f^{(2n)}(1) M'_n(z) + f^{(2n+1)}(0) \widetilde{M}'_n(z) \right)$$

and

$$\widetilde{f}(z) = f'(1-z) = \sum_{n=0}^{\infty} \left(f^{(2n)}(1) M'_n(1-z) + f^{(2n+1)}(0) \widetilde{M}'_n(1-z) \right).$$

The coefficient of $f^{(2n+2)}(1)$ is $M'_{n+1}(1-z)$.

However we also have

$$\widetilde{f}(z) = \sum_{n=0}^{\infty} \left(\widetilde{f}^{(2n)}(1) M_n(z) + \widetilde{f}^{(2n+1)}(0) \widetilde{M}_n(z) \right).$$

Since $\widetilde{f}^{(2n)}(1) = -f^{(2n+1)}(0)$ and $\widetilde{f}^{(2n+1)}(0) = -f^{(2n+2)}(1)$, this yields

$$\widetilde{f}(z) = \sum_{n=0}^{\infty} \left(-f^{(2n+1)}(0) M_n(z) - f^{(2n+2)}(1) \widetilde{M}_n(z) \right).$$

The coefficient of $f^{(2n+2)}(1)$ is $-\widetilde{M}_n(z)$.

From the unicity of the expansion we conclude

$$-\widetilde{M}_n(z) = M'_{n+1}(1-z)$$

for $n \geq 0$ (and $M'_0 = 0$).

7. Let S be a positive integer and let $z \in \mathbb{C}$. Using Cauchy's residue Theorem, compute the integral (see p. 39)

$$\frac{1}{2\pi i} \int_{|t|=S\pi} t^{-2n-1} \frac{\mathrm{ch}(tz)}{\mathrm{ch}(t)} \mathrm{d}t.$$

$\boxed{\mathbf{7.}}$ The poles of the function

$$t \mapsto \frac{\mathrm{ch}(tz)}{\mathrm{ch}(t)} = \frac{\mathrm{e}^{tz} + \mathrm{e}^{-tz}}{\mathrm{e}^{t} + \mathrm{e}^{-t}}$$

are the complex numbers t such that $\mathrm{e}^{2t} = -1$, namely $t = \left(s + \frac{1}{2}\right) i\pi$, $s \in \mathbb{Z}$.

The poles inside $|t| \leq S\pi$ are the numbers $\left(s + \frac{1}{2}\right) i\pi$ and $\left(-s - \frac{1}{2}\right) i\pi$ with $0 \leq s \leq S$.

The residue at $t = 0$ of $t^{-2n-1} \frac{\mathrm{ch}(tz)}{\mathrm{ch}(t)}$ is the coefficient ot t^{-2n} in the Taylor expansion of $\frac{\mathrm{ch}(tz)}{\mathrm{ch}(t)}$, hence it is $M_n(z)$.

Let s be an integer in the range $0 \leq s \leq S$. Write $t = \left(s + \frac{1}{2}\right) i\pi + \epsilon$. Then

$$\mathrm{e}^{t} = (-1)^s \mathrm{e}^{i\pi/2} e^{\epsilon} = (-1)^s i (1 + \epsilon + \cdots), \quad \mathrm{e}^{-t} = (-1)^s \mathrm{e}^{-i\pi/2} e^{\epsilon} = -(-1)^s i (-1)^s (1 - \epsilon + \cdots,)$$

$$\mathrm{e}^t + \mathrm{e}^{-t} = (-1)^s 2i\epsilon + \cdots,$$

and

$$\mathrm{e}^{tz} + \mathrm{e}^{-tz} = 2\cos\left(\frac{2s+1}{2}\pi z\right) + \cdots$$

Therefore, for $s \geq 0$, the residue $t = \left(s + \frac{1}{2}\right) i\pi$ of $t^{-2n-1}\frac{\mathrm{ch}(tz)}{\mathrm{ch}(t)}$ is

$$(-1)^{n+s}\left(s + \frac{1}{2}\right)^{-2n-1} \pi^{-2n-1} \cos\left(\frac{2s+1}{2}\pi z\right).$$

For $0 \leq s \leq S$, the residue at $\left(-s - \frac{1}{2}\right) i\pi$ is the same.

This proves the formula p. 39 :

$$M_n(z) = (-1)^n \frac{2^{2n+2}}{\pi^{2n+1}} \sum_{s=0}^{S-1} \frac{(-1)^s}{(2s+1)^{2n+1}} \cos\left(\frac{(2s+1)\pi}{2} z\right) + \frac{1}{2\pi i}\int_{|t|=S\pi} t^{-2n-1}\frac{\mathrm{ch}(tz)}{\mathrm{ch}(t)}\mathrm{d}t$$

for $S = 1, 2, \ldots$ and $z \in \mathbb{C}$.

8. Give examples of complete, redundant and indeterminate systems in Whittaker classification p. 43.

$\boxed{8.}$

• Complementary sequences (each integer belongs to one and only one of the two sets) are complete. For instance the set of two sequences

$$(1, 3, 5, \ldots, 2n+1, \ldots), \quad (0, 2, 4, \ldots, 2n, \ldots)$$

is complete (Whittaker).

• The set of two sequences

$$(0, 2, 4 \ldots, 2n, \ldots), \quad (0, 2, 4 \ldots, 2n, \ldots)$$

is complete (Lidstone).

• The set of two sequences

$$(1, 3, 5, \ldots, 2n+1, \ldots), \quad (1, 3, 5, \ldots, 2n+1, \ldots)$$

is indeterminate (more than one solution to the interpolation problem). If one adds 0 to one set,

$$(0, 1, 3, 5, \ldots, 2n+1, \ldots), \quad (1, 3, 5, \ldots, 2n+1, \ldots)$$

one gets a complete set.

• Given any sequence $(q_0, q_1, q_2, \ldots)$, the set of two sequences

$$(0, 1, 2, \ldots, n, \ldots), \quad (q_0, q_1, q_2, \ldots)$$

is redundant (no solution to the interpolation problem).

• The set of two sequences

$$(0, 2, 4, 6, 8, \ldots, 2n, \ldots), \quad (0, 1, 3, 5, \ldots, 2n+1, \ldots)$$

is redundant (no solution to the interpolation problem).

• According to [Whittaker, 1933], a pair of sequences $(p_0, p_1, p_2, \dots)$, $(q_0, q_1, q_2, \dots)$ is complete if and only if the sequence $(D(1), D(2), D(3), \dots)$, defined by

$$D(m) \text{ is the number of } p \text{ and } q \text{ which are } < m$$

satisfies

$$D(m) \geq m \text{ for all } m \geq 1 \text{ and } D(m) = m \text{ for infinitely many } m.$$

Given a complete pair of sequences, if we remove some elements, we get an indeterminate pair. Given an indeterminate pair of sequences, it is possible to add some elements and get a complete pair.

Références

[Whittaker, 1933] Whittaker, J. M. (1933). On Lidstone's series and two-point expansions of analytic functions. *Proc. Lond. Math. Soc. (2)*, 36 :451–469.
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