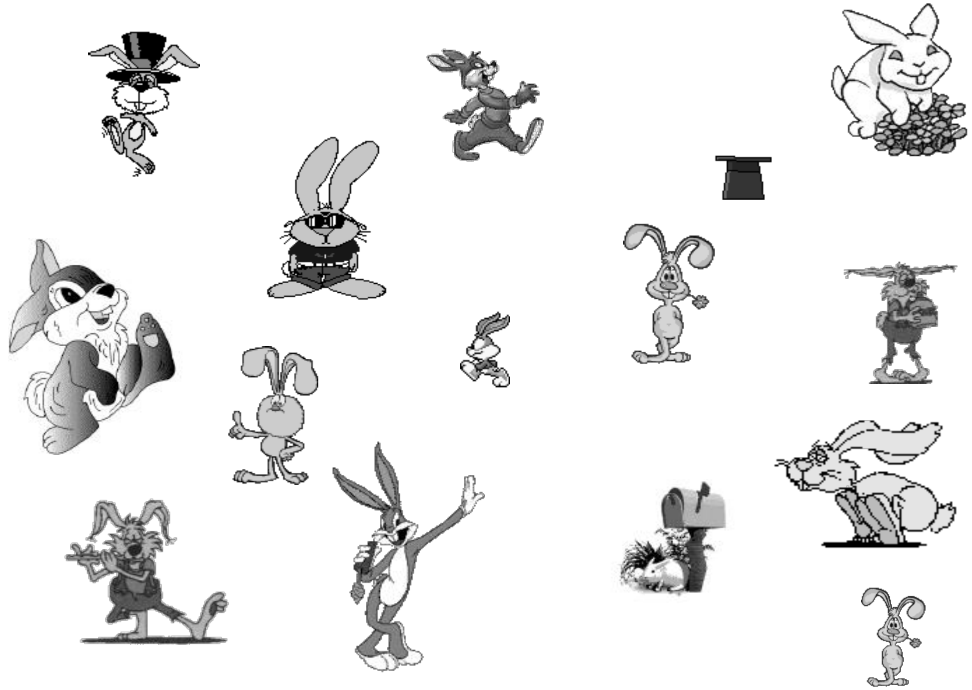
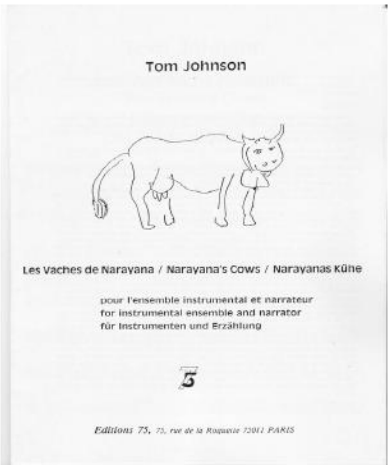
Forman Christian College (FC College) University Lahore - November 26, 2015

Some arithmetic problems raised by rabbits, cows and the Da Vinci Code

Michel Waldschmidt
Université P. et M. Curie (Paris VI)

http://www.imj-prg.fr/~michel.waldschmidt/





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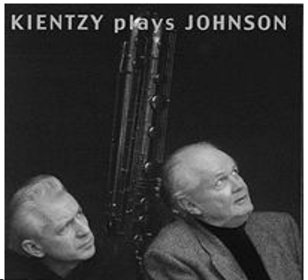


Narayana's Cows

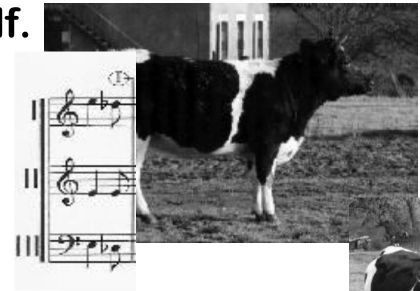
Music: Tom Johnson
Saxophones: Daniel Kientzy
Realization: Michel Waldschmidt

http://www.math.jussieu.fr/~miw/

Narayana was an Indian mathematician in the 14th. century, who proposed the following problem:
A cow produces one calf every year.
Begining in its fourth year, each calf produces one calf at the begining of each year.
How many cows are there altogether after, for example, 17 years?

While you are working on that,
let us give you a musical demonstration.

The first year there is only the original cow and her first calf.



Year	1
Original Cow	1
Second generation	1
Total	**2**

long-short

The second year there is the original cow and 2 calves.

Year	1	2
Original Cow	1	1
Second generation	1	+ 2
Total	2	= 3

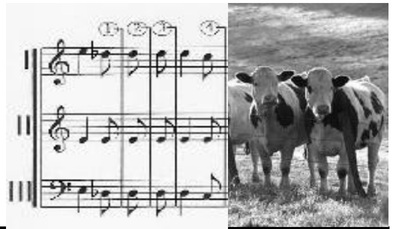
long-short -short

The third year there is the original cow and 3 calves.

Year	1	2	3
Original Cow	1	1	1
Second generation	1	2	+ 3
Total	**2**	**3**	= **4**

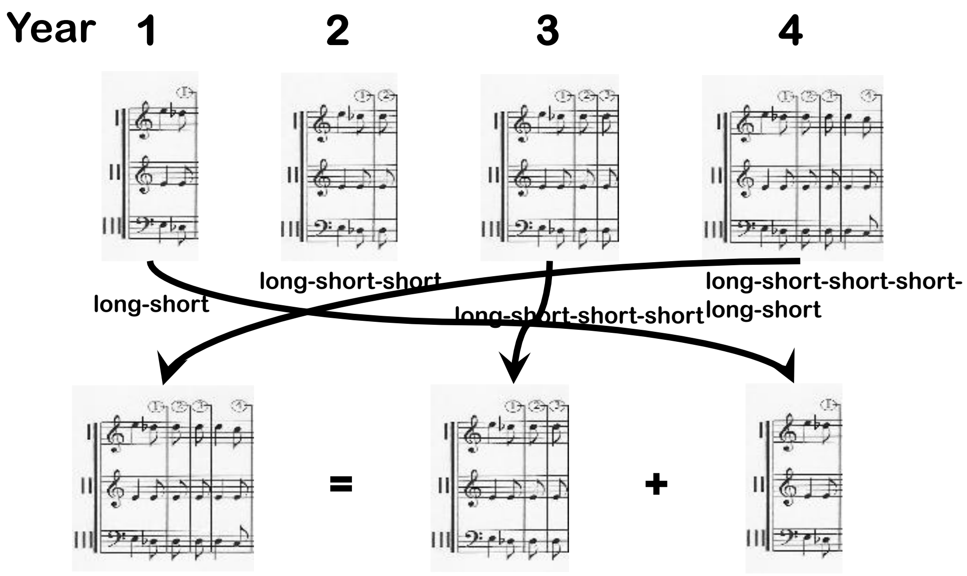
long-short -short -short

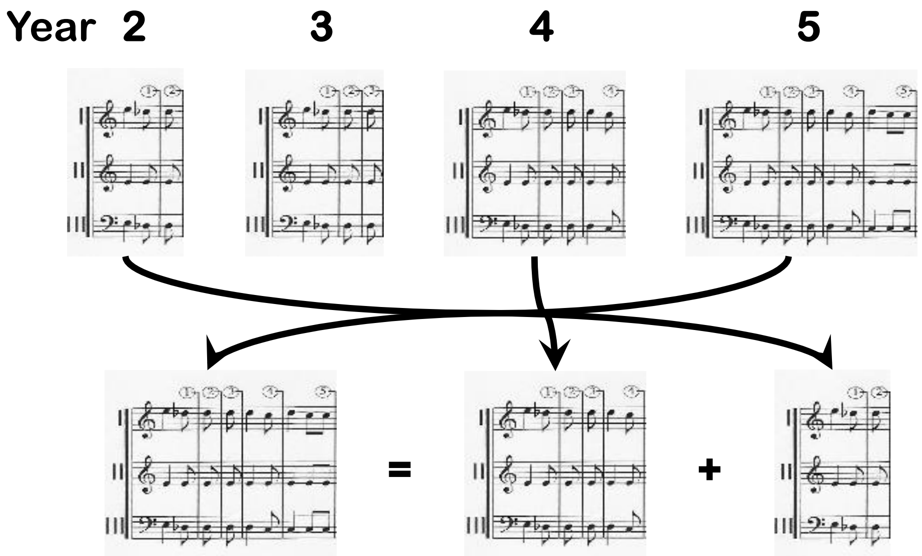
The fourth year the oldest calf becomes a mother, and we begin a third generation of Naryana's cows.



Year	1	2	3	4
Original Cow	1	1	1	1
Second generation	1	2	3	4
Third generation	0	0	0	1
Total	**2**	**3**	**4**	**6**

long - short - short - short - long - short





The fifth year we have another mother cow and 3 new calves.

Year	1	2	3	4	5	
Original Cow	1	1	1	1	1	
Second generation	1	2	3	4	5	+1
Third generation	0	0	0	1	3 +	+2
Total	**2**	**3**	**4**	**6**	**9** =	+3

The sixth year we have 4 productive cows, 4 new calves, and a total herd of 13.

Year	1	2	3	4	5	6
Original Cow	1	1	1	1	1	1
Second generation	1	2	3	4	5	6
Third generation	0	0	0	1 +	3	6
Total	**2**	**3**	**4**	**6**	**9**	**13**

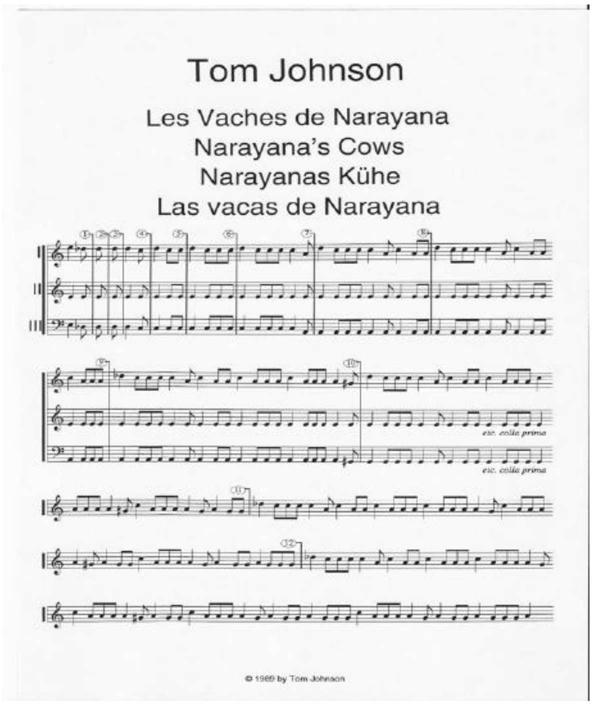
The sixth year

4 productive cows = 4 long

9 young calves = 9 short

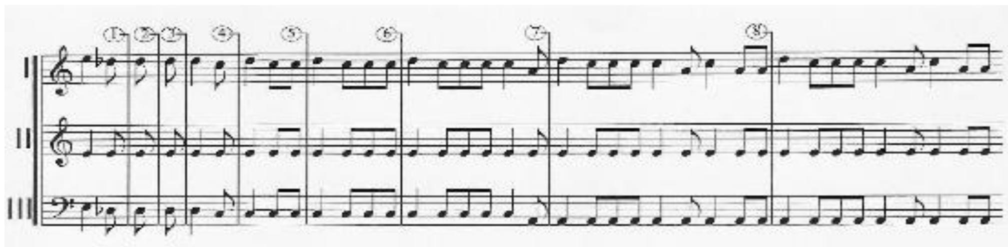
Total: 13 cows = 13 notes

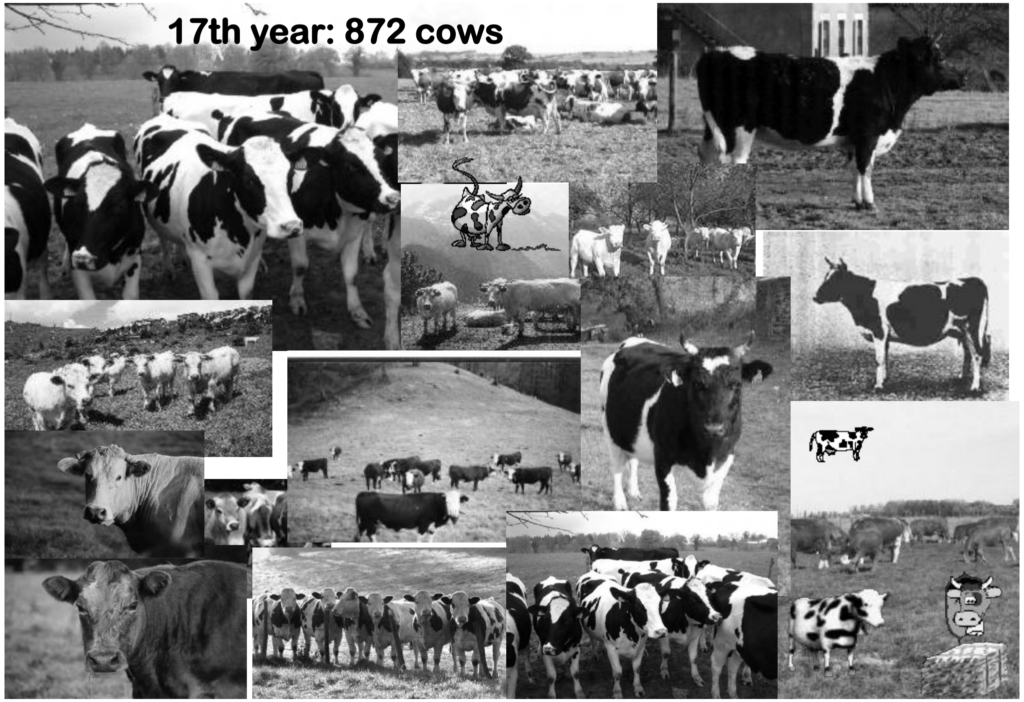
Year	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Original Cow	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Second generation	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Third generation	0	0	0	1	3	6	10	15	21	28	36	45	55	66	78	91	105
Fourth generation	0	0	0	0	0	0	1	4	10	20	35	56	84	120	165	220	286
Fifth generation	0	0	0	0	0	0	0	0	0	1	5	15	35	70	126	210	330
Sixth generation	0	0	0	0	0	0	0	0	0	0	0	0	1	6	21	56	126
Seventh generation	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	7
Total	**2**	**3**	**4**	**6**	**9**	**13**	**19**	**28**	**41**	**60**	**88**	**129**	**189**	**277**	**406**	**595**	**872**



Tom Johnson

Les Vaches de Narayana
Narayana's Cows





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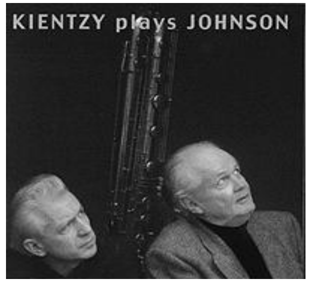

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Total	**2**

long-short

The second year there is the original cow and 2 calves.

Year	1	2
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Total	**2**	**3**

long -short -short

The third year there is the original cow and 3 calves.

Year	1	2	3
Original Cow	1	1	1
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Total	**2**	**3**	**4**

long -short -short -short

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long - short - short - short - long - short

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Year	1	2	3	4	5	
Original Cow	1	1	1	1	1	
Second generation	1	2	3	4	5	+1
Third generation	0	0	0	1	3	+2
Total	**2**	**3**	**4**	**6**	**9**	+3

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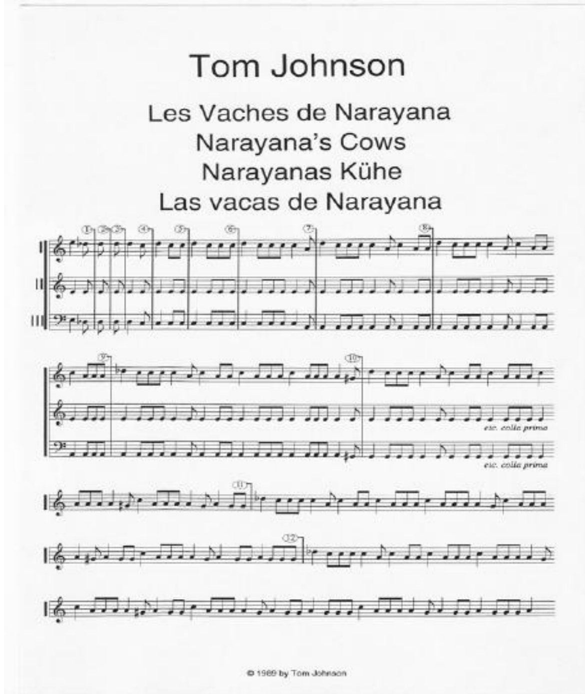
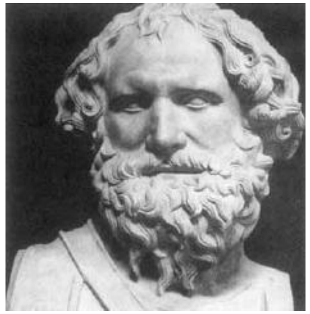
Year	1	2	3	4	5	6
Original Cow	1	1	1	1	1	1
Second generation	1	2	3	4	5	6
Third generation	0	0	0	1	3	6
Total	**2**	**3**	**4**	**6**	**9**	**13**

Archimedes cattle problem



Archimedes



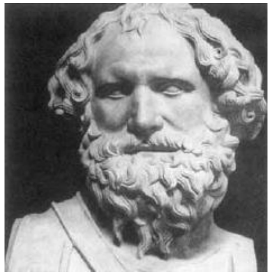
The cattle problem of Archimedes asks to determine the size of the herd of the God Sun.

The sun god had a herd of cattle consisting of bulls and cows, one part of which was white, a second black, a third spotted, and a fourth brown.
Among the bulls, the number of white ones was one half plus one third the number of the black greater than the brown.

Archimedes Cattle Problem

The number of the black, one quarter plus one fifth the number of the spotted greater than the brown.
The number of the spotted, one sixth and one seventh the number of the white greater than the brown.

Among the cows, the number of white ones was one third plus one quarter of the total black cattle. The number of the black, one quarter plus one fifth the total of the spotted cattle; The number of spotted, one fifth plus one sixth the total of the brown cattle



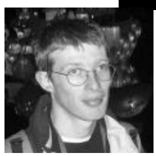
Archimedes cattle problem

There are infinitely many solutions.

The smallest one has *206 545* digits.

This problem was almost solved by a german mathematician, A. Amthor, in 1880, who commented: « *Assume that the size of each animal is less than the size of the smallest bacteria. Take a sphere of the same diameter as the size of the milked way, which the light takes ten thousand years to cross. Then this sphere would contain only a tiny proportion of the herd of the God Sun.* »

Number of atoms in the known finite universe



When I was young: 10^{60} atoms
(1 followed by 60 zeroes)

A few years later (long back): 10^{70}

Nowadays: ?

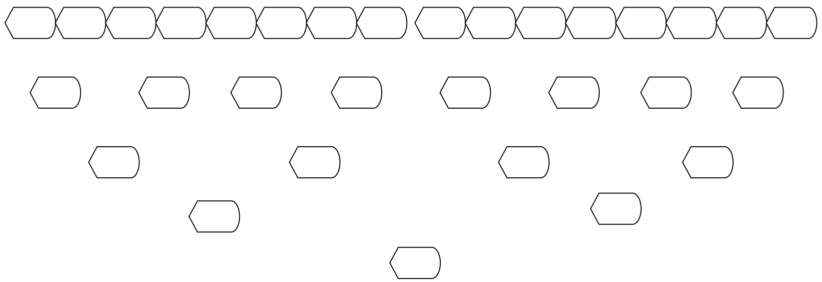
Solution of Archimedes Problem

A. Amthor
"Das Problema bovinum des Archimedes »
Zeitschrift fur Math. u. Physik. (Hist.-litt.Abtheilung) Volume **XXV** (1880), pages 153-171

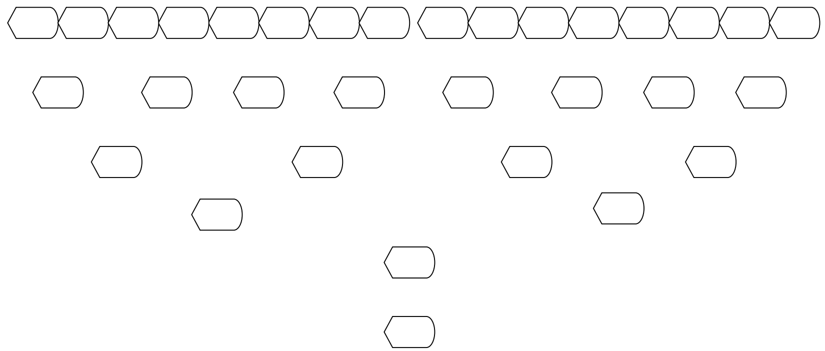
H.C. Williams, R.A. German and C.R. Zarnke, *Solution of the cattle problem of Archimedes*, Math. Comp., **19**, 671-674 (1965).

How many ancesters do we have?

Sequence: 1, 2, 4, 8, 16 …



Bees genealogy



Bees genealogy

Number of females at a given level =
total population at the previous level

Number of males at a given level=
number of females at the previous level

Sequence: 1, 1, 2, 3, 5, 8, …

3 + 5 = 8

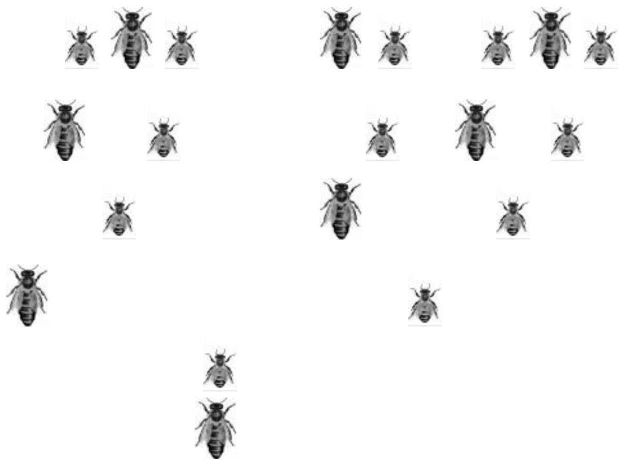
2 + 3 = 5

1 + 2 = 3

1 + 1 = 2

0 + 1 = 1

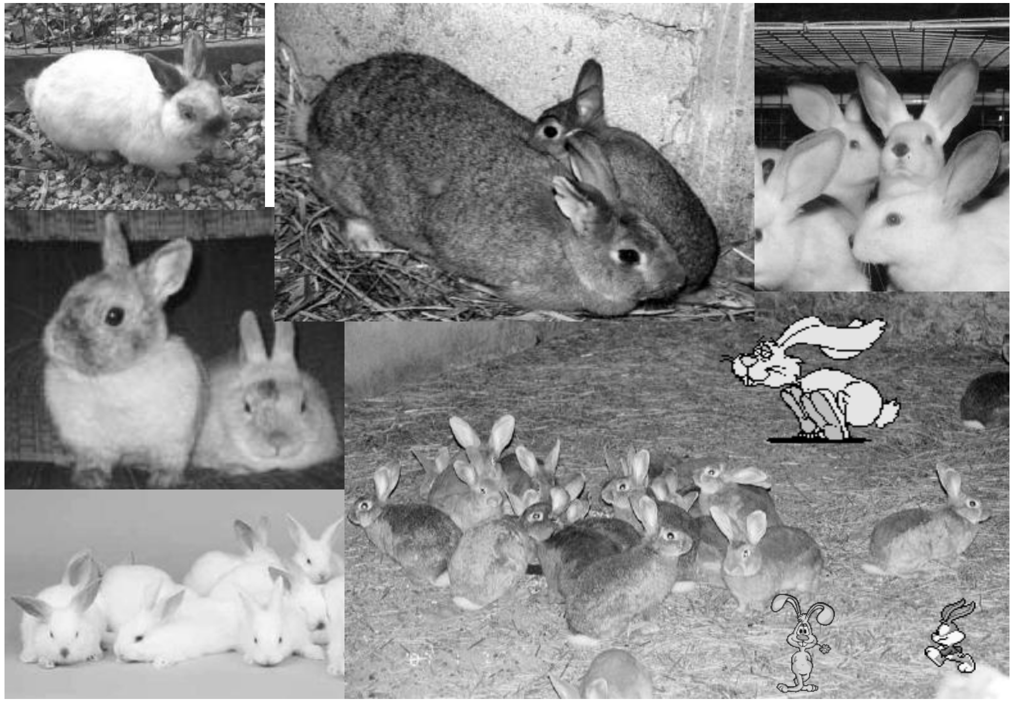
1 + 0 = 1



Fibonacci (Leonardo di Pisa)

- Pisa (Italia)
  ≈ 1175 - 1250
- Liber Abaci ≈ 1202
- 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55,…
- Each term is the sum of the two preceding ones: 21+34=55

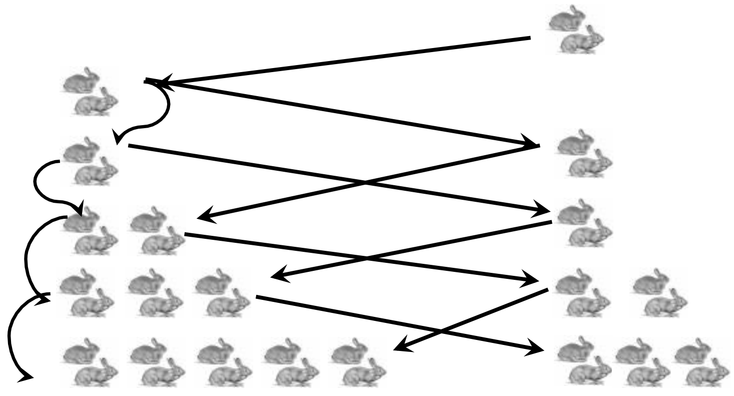
The Fibonacci Quarterly
Official Publication of The Fibonacci Association



Modelization of a population

Adult pairs **Young pairs**

- First month
- Second month
- Third month
- Fourth month
- Fifth month
- Sixth month



Sequence: 1, 1, 2, 3, 5, 8, …

The Fibonacci sequence

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610,

1+1=2

1+2=3

2+3=5

3+5=8

5+8=13

8+13=21

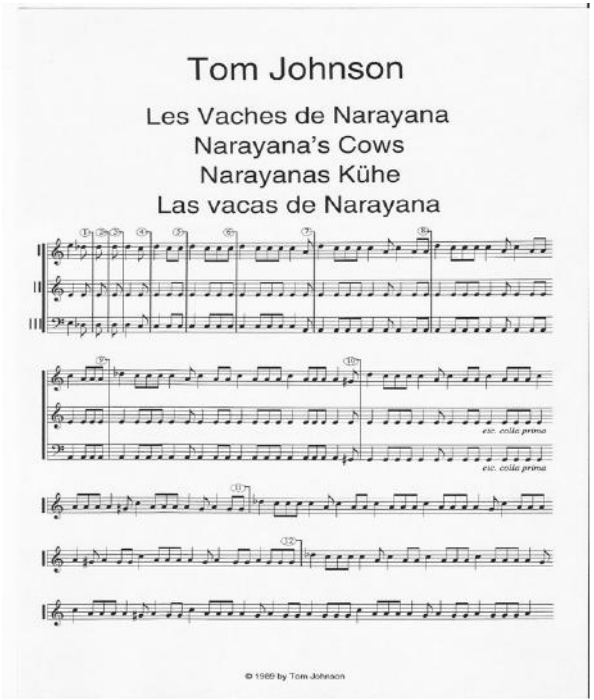
13+21=34

21+34=55

Narayana cows (back)

année	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Total	**2**	**3**	**4**	**6**	**9**	**13**	**19**	**28**	**41**	**60**	**88**	**129**	**189**	**277**	**406**	**595**	**872**

Each term is the sum of the preceding one and the one before the penultimate:
872=595+277



Theory of stable populations (Alfred Lotka)

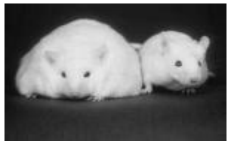
Assume each pair generates a new pair the first two years only. Then the number of pairs who are born each year again follow the Fibonacci rule.

Arctic trees

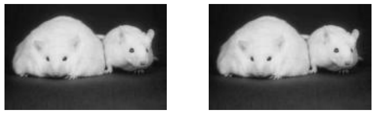
In cold countries, each branch of some trees gives rise to another one after the second year of existence only.

Exponential sequence

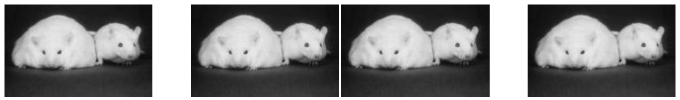
- First month



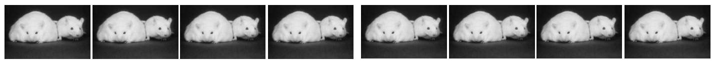
- Second month



- Third month



- Fourth month



Number of pairs: 1, 2, 4, 8, …

Size of Narayana's herd after 60 years:

11 990 037 126 *(11 digits)*

Number of Fibonacci rabbits after 60 months:

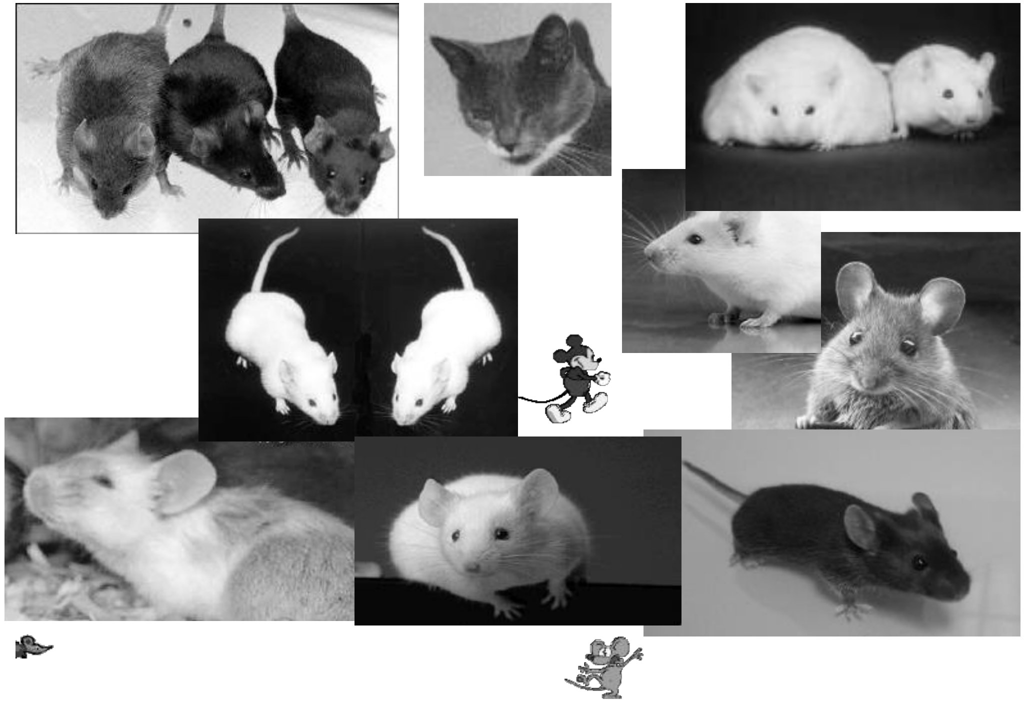
1 548 008 755 920 *(13 digits)*

Number of pairs of mice after 60 months:

1 152 921 504 606 846 976 *(19 digits)*

Exponential growth

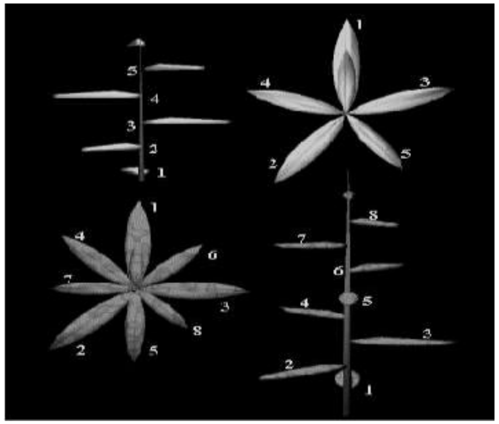
- Beetle larvas
- Bacteria
- Economy



Phyllotaxy

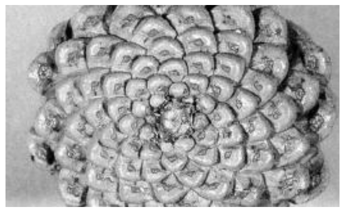
- Study of the position of leaves on a stem and the reason for them
- Number of petals of flowers: daisies, sunflowers, aster, chicory, asteraceae,…
- Spiral patern to permit optimal exposure to sunlight
- Pine-cone, pineapple, Romanesco cawliflower, cactus

Leaf arrangements



- Université de Nice,

  Laboratoire Environnement Marin Littoral, Equipe d'Accueil "Gestion de la Biodiversité"



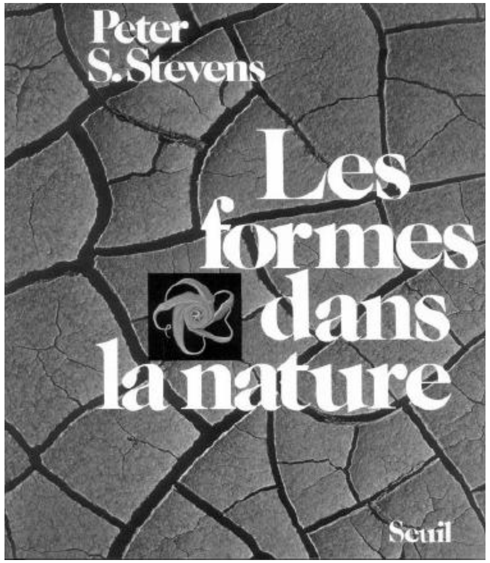
http://www.unice.fr/LEML/coursJDV/tp/tp3.htm

Phyllotaxy




Phyllotaxy

- J. Kepler (1611) uses the Fibonacci sequence in his study of the dodecahedron and the icosaedron, and then of the symmetry of order 5 of the flowers.
- Stéphane Douady and Yves Couder
  Les spirales végétales
  La Recherche *250* (Jan. 1993) vol. **24.**



En effeuillant la marguerite / Plucking the daisy petals

I love you / *He loves me*
A little bit / *He loves me not*
A lot / *Beaucoup*
With passion / *Passionnément*
With madness / *A la folie*
Not at all / *Pas du tout*

Sequence of the remainders of the division by 6 of the Fibonacci numbers

1, 1, 2, 3, 5, 2, 1, 3, 4, 1, 5, 0, 5,…

First multiple of 6 : 144

Division by 6 of the Fibonacci numbers

$8 = 6 \times 1 + 2$

$13 = 6 \times 2 + 1$

$21 = 6 \times 3 + 3$

$34 = 6 \times 5 + 4$

$55 = 6 \times 9 + 1$

$89 = 6 \times 14 + 5$

$144 = 6 \times 24 + 0$

I love you:	+1
A little bit :	+2
A lot :	+3
With passion :	+4
With madness :	+5
Not at all :	+0

Division by 2 of the Fibonacci numbers

$8 = 2 \times 4 + 0$

$13 = 2 \times 6 + 1$

$21 = 2 \times 10 + 1$

$34 = 2 \times 17 + 0$

$55 = 2 \times 27 + 1$

$89 = 2 \times 44 + 1$

$144 = 2 \times 72 + 0$

He loves me :	+1
He loves me not	+0

The Da Vinci Code

Five enigmas to be solved

In the book written by Dan Brown in 2003 one finds some (weak) crypto techniques.

1

The first enigma asks for putting in the right order the integers of the sequence

1 3 - 3 - 2 - 2 1 - 1 - 1 - 8 - 5.

This reordering will provide the key of the bank account.

2

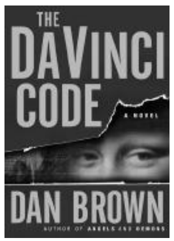
An english anagram

O DRACONIAN DEVIL, OH LAME SAINT

3

A french anagram

SA CROIX GRAVE L'HEURE



The Da Vinci Code

Five enigmas to be solved (continued)

4

A french poem to be decoded :

élc al tse essegas ed tom xueiv nu snad
eétalcé ellimaf as tinuér iuq
sreilpmet sel rap éinéb etêt al
eélévér ares suov hsabta ceva

5

An old wisdom word to be found.

Answer for* 5*:* SOPHIA *(Sophie Neveu)

The Da Vinci Code

the bank account key involving eight numbers

The eight numbers of the key of the bank account are:

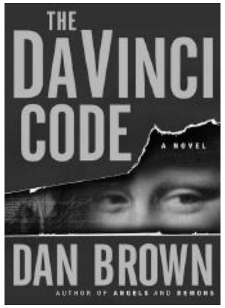
13 - 3 - 2 - 21 - 1 - 1 - 8 - 5

These are the eight first integers of the Fibonacci sequence.
The goal is to find the right order at the first attempt. The right answer is given by selecting the natural ordering:

1 - 1 - 2 - 3 - 5 - 8 - 13 - 21

The total number of solutions is

20 160



Primitive languages

Given some letters, how many words does one obtain
if one uses each letter exactly once?

With 1 letter a, there is just one word: a.

With 2 letters a,b, there are 2 words, namely

ab, ba.

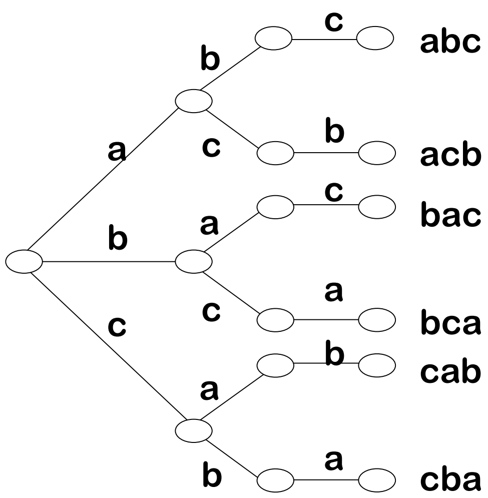
With 3 letters a,b,c : select the first letter (3 choices), once it is selected, complete with the 2 words involving the 2 remaining letters. Hence the number of words is $3 \cdot 2 \cdot 1=6$, namely

abc, acb,
bac, bca,
cab, cba.

Three letters: a, b, c **First letter Second Third Word**

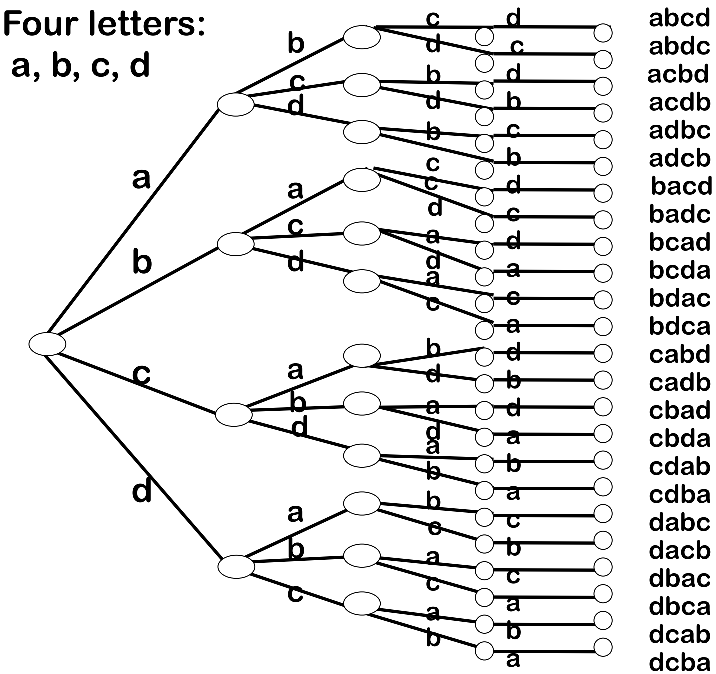
Six words

$3 \cdot 2 \cdot 1=6$



3 2 1

Four letters:
a, b, c, d



$4 \cdot 3 \cdot 2 \cdot 1$
$=24$

The sequence **1 3 - 3 - 2 - 2 1 - 1 - 1 - 8 - 5**

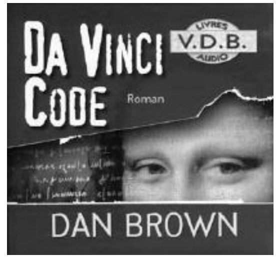
In the same way, with 8 letters, the number of words is

8· 7 ·6 ·5 ·4 ·3 ·2 ·1= 40 320.

Here the digit 1 occurs twice, this is why the number of orderings is only half:

20 160

The Da Vinci Code



2

An english anagram

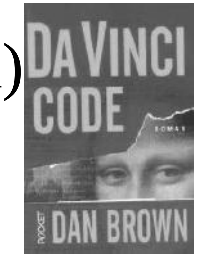
DRACONIAN DEVIL, OH LAME SAINT
THE MONA LISA LEONARDO DA VINCI

3

A french anagram

SA CROIX GRAVE L'HEURE
LA VIERGE AUX ROCHERS

The Da Vinci Code (continued)

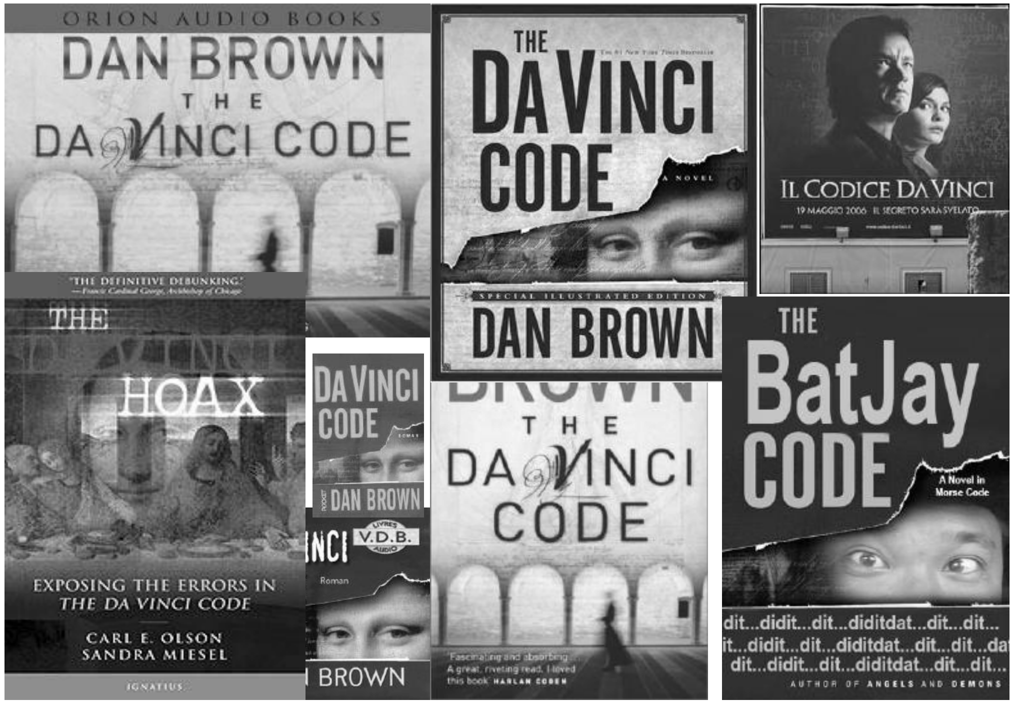


4

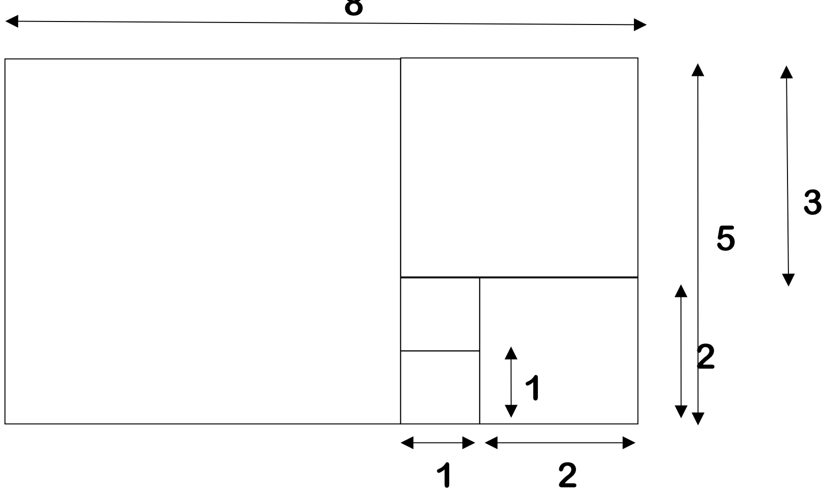
A french poem to decode:
élc al tse essegasedtom xueiv nu snad
eétalcé ellimaf as tinuér iuq
sreilpmet sel rap éinéb etêt al
eélévér ares suov hsabta ceva

dans un vieux mot de sagesse est la clé
qui réunit sa famille éclatée
la tête bénie par les Templiers
avec Atbash vous sera révélée

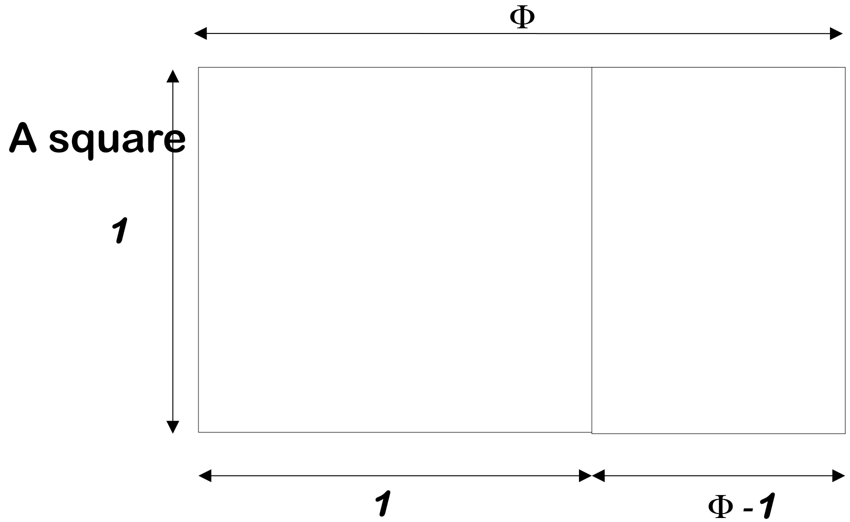
« utiliser un miroir pour déchiffrer le code »
« use a mirror for decoding»



Geometric construction of the Fibonacci sequence

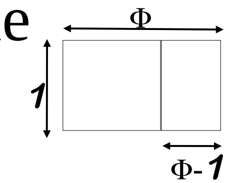


This is a nice rectangle



A NICE RECTANGLE

Golden Rectangle



Sides: **1** *and* Φ

Condition:

The two rectangles with sides

1 *and* Φ *for the big one,*

Φ-**1** *and* **1** *for the small one,*

have the same proportions.

Proportion of the big one: $\dfrac{\Phi}{1} = \Phi$

Proportion of the small one: $\dfrac{1}{\Phi - 1}$

Golden Rectangle

Sides: **1** *and* $\Phi = 1.618033\ldots$

Condition: $\Phi = \dfrac{1}{\Phi - 1}$ $\qquad \Phi(\Phi - 1) = 1$

Equation: $\Phi^2 - \Phi - 1 = 0$ $\qquad \Phi - 1 = \dfrac{1}{\Phi}$

Solution:

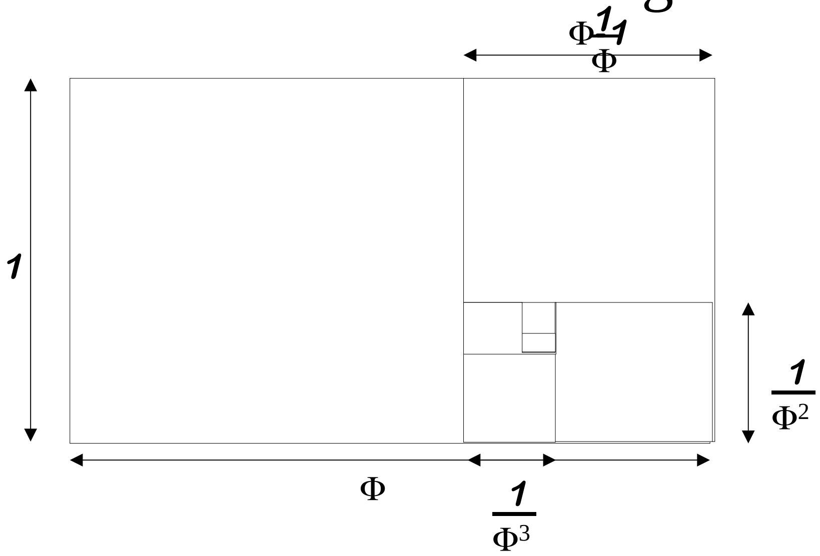
Φ *is the* ***Golden Number***

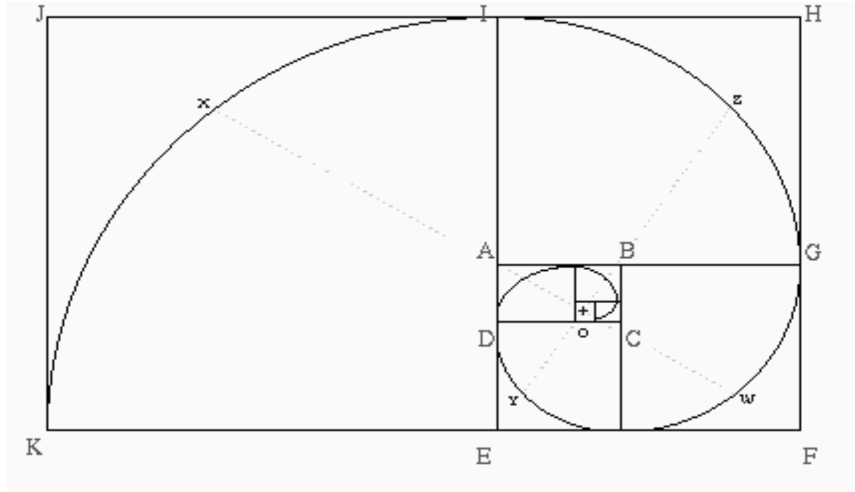
1.618033…

To go from the large rectangle to the small one: divide each side by Φ

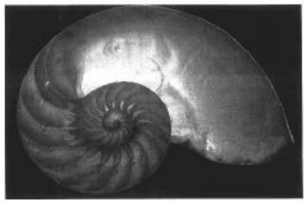
$$\Phi - 1 = \frac{1}{\Phi}$$

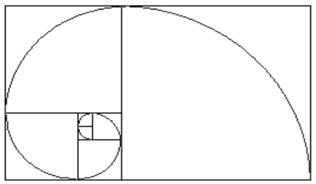
The Golden Rectangle

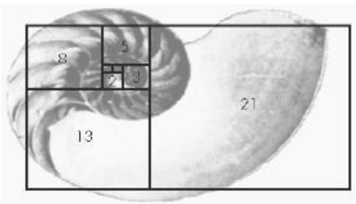


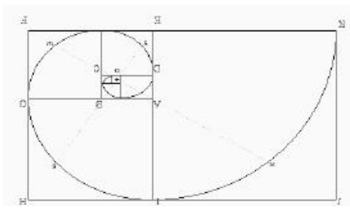


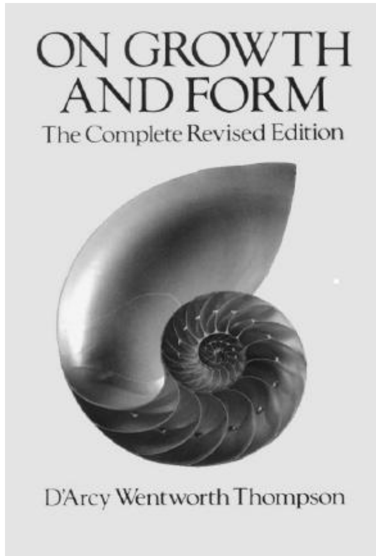
Ammonite (Nautilus shape)



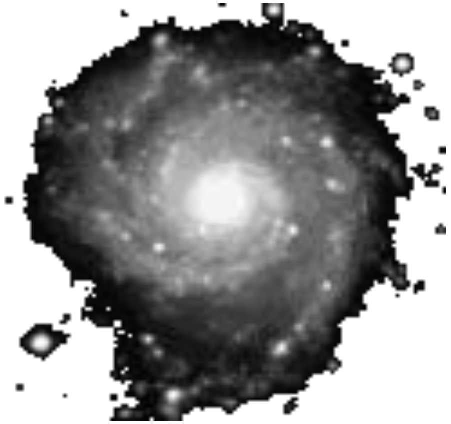




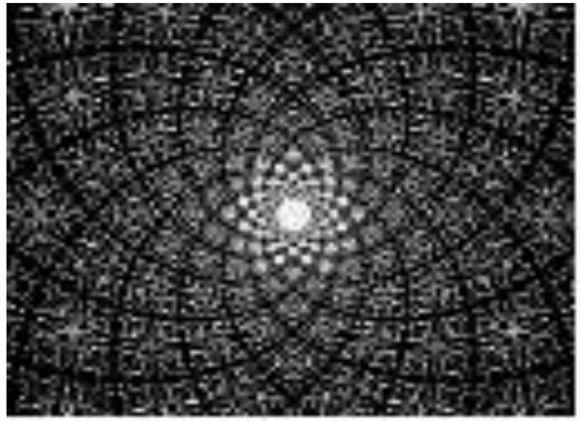




Spirals in the Galaxy

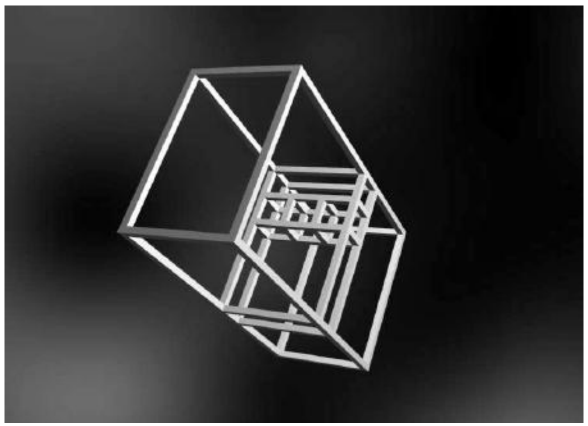


The Golden Number in art, architecture,… aesthetics

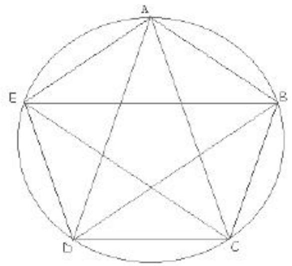


Kees van Prooijen

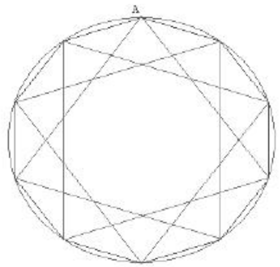
http://www.kees.cc/gldsec.html



Regular pentagons and dodecagons



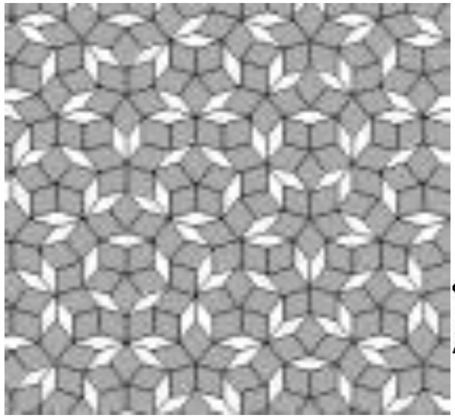
Pentagones réguliers

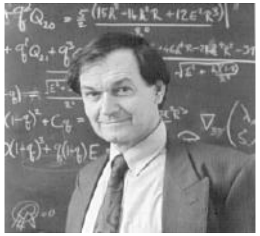


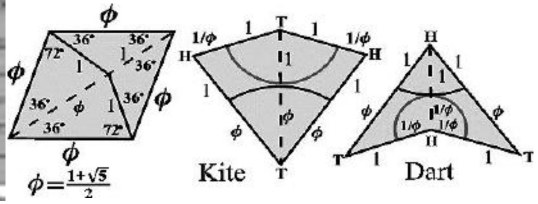
Les décagones réguliers

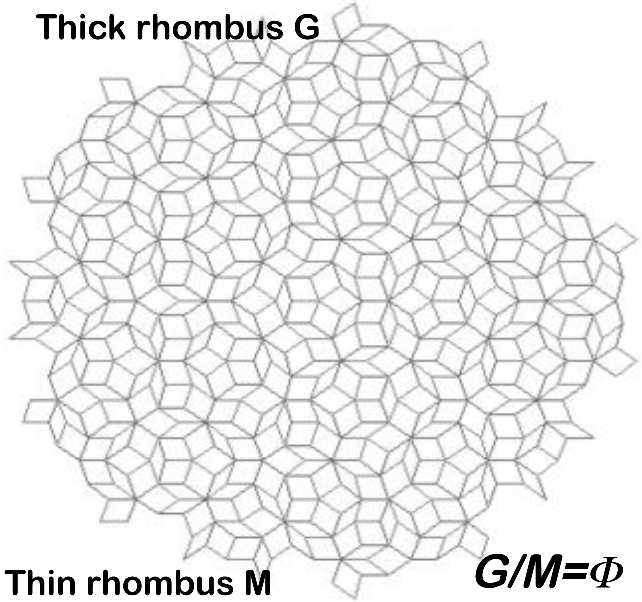
$\Phi = 2 \cos(\pi/5)$

Penrose non-periodic tiling patterns and quasi-crystals

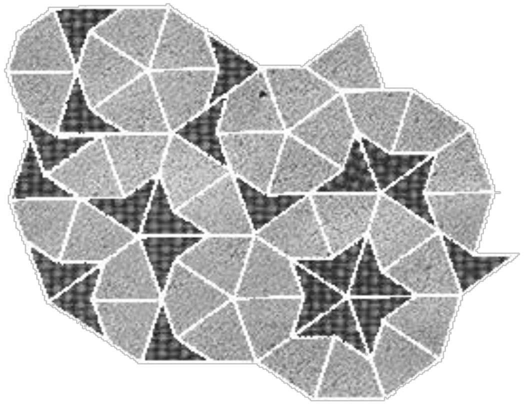




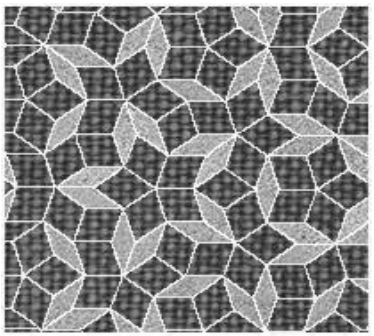




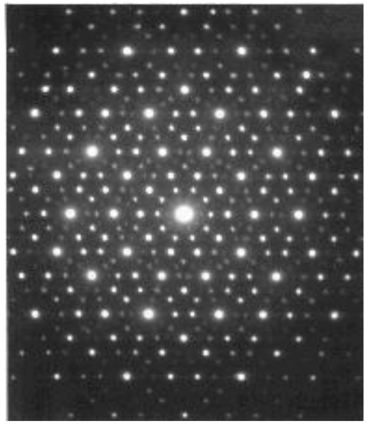
G/M=Φ



proportion=Φ



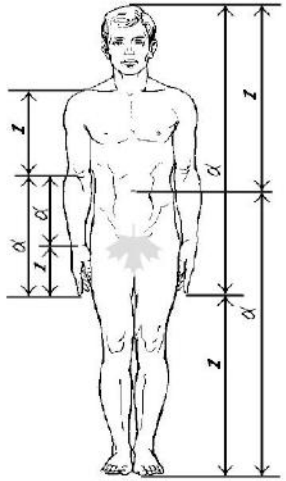
Diffraction of quasi-crystals



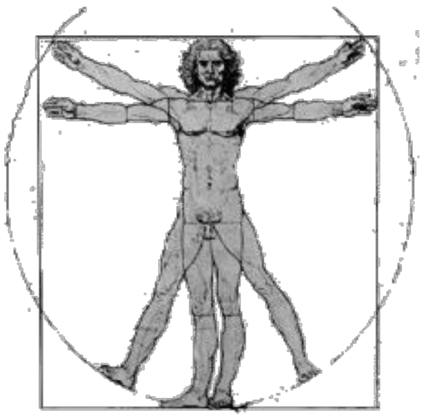
Doubly periodic tessalation (lattices) - cristallography

Géométrie d'un champ de lavande
http://math.unice.fr/~frou/lavande.html
François Rouvière (Nice)

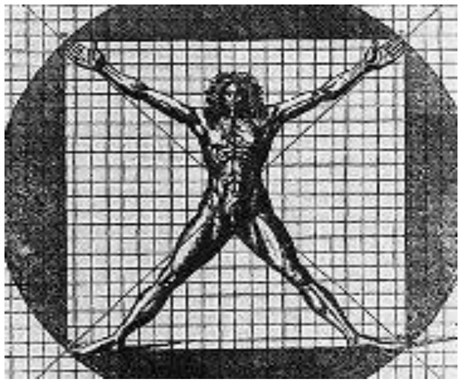
The Golden Number and aesthetics



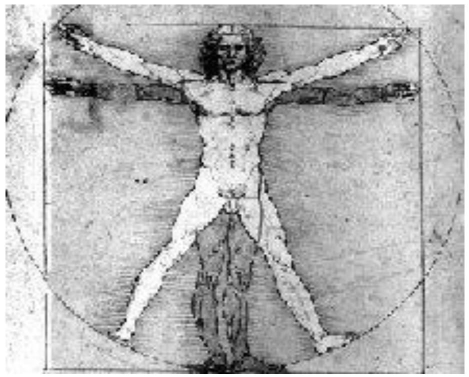
The Golden Number and aesthetics



Marcus Vitruvius Pollis (Vitruve, 88-26 av. J.C.)



Léonard de Vinci (Leonardo da Vinci, 1452-1519)



Music and the Fibonacci sequence

- Dufay, XV^{ème} siècle
- Roland de Lassus
- Debussy, Bartok, Ravel, Webern
- Stockhausen
- Xenakis
- **Tom Johnson** *Automatic Music for six percussionists*

Example of a recent result

Y. Bugeaud, M. Mignotte, S. Siksek (2004):
The only perfect powers (squares, cubes, etc.) in the Fibonacci sequence are 1, 8 and 144.

The quest of the Graal for a mathematician

Open problems, conjectures.

Example of an unsolved question:
Are there infinitely many primes in the Fibonacci sequence?

Some applications of Number Theory

- **Cryptography, security of computer systems**
- **Data transmission, error correcting codes**
- **Interface with theoretical physics**
- **Musical scales**
- **Numbers in nature**

Some arithmetic problems raised by rabbits, cows and the Da Vinci Code

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Université P. et M. Curie (Paris VI)

http://www.imj-prg.fr/~michel.waldschmidt/