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Institute of Mathematics of  
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# Transcendental Number Theory: recent results and open problems.

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## **Travel Grants for 1,000 Mathematicians ICM 2014 Invitation Program : “NANUM 2014”**

<http://www.icm2014.org/>

Tentative schedule for the application and review process :

- Call for application : June 1, 2013
- Deadline to submit all the application documents : August 31, 2013
- Selection of the travel grant recipient : September 2013  
December 2013
- Notification to applicants of acceptance : January 2014

# Abstract

An algebraic number is a complex number which is a root of a polynomial with rational coefficients. For instance  $\sqrt{2}$ ,  $i = \sqrt{-1}$ , the Golden Ratio  $(1 + \sqrt{5})/2$ , the roots of unity  $e^{2i\pi a/b}$ , the roots of the polynomial  $X^5 - 6X + 3$  are algebraic numbers. A **transcendental number** is a complex number which is not algebraic.

## Abstract (continued)

The existence of transcendental numbers was proved in 1844 by J. Liouville who gave explicit ad-hoc examples. The transcendence of constants from analysis is harder; the first result was achieved in 1873 by Ch. Hermite who proved the transcendence of  $e$ . In 1882, the proof by F. Lindemann of the transcendence of  $\pi$  gave the final (and negative) answer to the Greek problem of squaring the circle. The transcendence of  $2^{\sqrt{2}}$  and  $e^{\pi}$ , which was included in Hilbert's seventh problem in 1900, was proved by Gel'fond and Schneider in 1934. During the last century, this theory has been extensively developed, and these developments gave rise to a number of deep applications. In spite of that, most questions are still open. In this lecture we survey the state of the art on known results and open problems.

<http://www.math.jussieu.fr/~miw/>

# Rational, algebraic irrational, transcendental

**Goal** : decide upon the arithmetic nature of “given” numbers :  
rational, algebraic irrational, transcendental.

Rational integers :  $\mathbf{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$ .

Rational numbers :

$$\mathbf{Q} = \{p/q \mid p \in \mathbf{Z}, q \in \mathbf{Z}, q > 0, \gcd(p, q) = 1\}.$$

**Algebraic number** : root of a polynomial with rational coefficients.

A **transcendental number** is complex number which is not algebraic.

# Rational, algebraic irrational, transcendental

**Goal** : decide whether a “given” real number is rational, algebraic irrational or else transcendental.

- **Question** : what means “given” ?
- Criteria for irrationality : development in a given basis (e.g. : decimal expansion, binary expansion), continued fraction.
- Analytic formulae, limits, sums, integrals, infinite products, any limiting process.

# Algebraic irrational numbers

Examples of algebraic irrational numbers :

- $\sqrt{2}$ ,  $i = \sqrt{-1}$ , the Golden Ratio  $(1 + \sqrt{5})/2$ ,
- $\sqrt{d}$  for  $d \in \mathbf{Z}$  not the square of an integer (hence not the square of a rational number),
- the roots of unity  $e^{2i\pi a/b}$ , for  $a/b \in \mathbf{Q}$ ,
- and, of course, any root of an irreducible polynomial with rational coefficients of degree  $> 1$ .

# Rule and compass ; squaring the circle

Construct a square with the same area as a given circle by using only a finite number of steps with compass and straightedge.

Any constructible length is an algebraic number, though not every algebraic number is constructible (for example  $\sqrt[3]{2}$  is not constructible).

Pierre Laurent Wantzel (1814 – 1848)

*Recherches sur les moyens de reconnaître si un problème de géométrie peut se résoudre avec la règle et le compas.* Journal de Mathématiques Pures et Appliquées **1** (2), (1837), 366–372.



# Quadrature of the circle

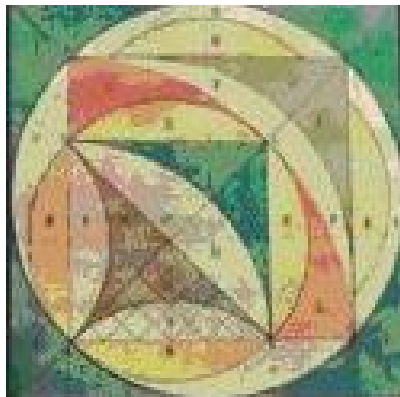
Marie Jacob

*La quadrature du cercle*

*Un problème*

*à la mesure des Lumières*

Fayard (2006).



# Resolution of equations by radicals

The roots of the polynomial  $X^5 - 6X + 3$  are algebraic numbers, and are not expressible by radicals.



Evariste Galois  
(1811 – 1832)

Born 200 years ago.

# Gottfried Wilhelm Leibniz

Introduction of the concept of the transcendental in mathematics by Gottfried Wilhelm Leibniz in 1684 :

*“Nova methodus pro maximis et minimis itemque tangentibus, qua nec fractas, nec irrationales quantitates moratur, . . .”*



Breger, Herbert. *Leibniz' Einführung des Transzendenten*, 300 Jahre “Nova Methodus” von G. W. Leibniz (1684-1984), p. 119-32. Franz Steiner Verlag (1986).

Serfati, Michel. *Quadrature du cercle, fractions continues et autres contes*, Editions APMEP, Paris (1992).

# §1 Irrationality

Given a basis  $b \geq 2$ , a real number  $x$  is rational if and only if its expansion in basis  $b$  is ultimately periodic.

$b = 2$  : binary expansion.

$b = 10$  : decimal expansion.

For instance the decimal number

0,123456789012345678901234567890 ...

is rational :

$$= \frac{1\ 234\ 567\ 890}{9\ 999\ 999\ 999} = \frac{137\ 174\ 210}{1\ 111\ 111\ 111}.$$

# First decimal digits of $\sqrt{2}$

<http://wims.unice.fr/wims/wims.cgi>

1.41421356237309504880168872420969807856967187537694807317667973  
799073247846210703885038753432764157273501384623091229702492483  
605585073721264412149709993583141322266592750559275579995050115  
278206057147010955997160597027453459686201472851741864088919860  
955232923048430871432145083976260362799525140798968725339654633  
180882964062061525835239505474575028775996172983557522033753185  
701135437460340849884716038689997069900481503054402779031645424  
782306849293691862158057846311159666871301301561856898723723528  
850926486124949771542183342042856860601468247207714358548741556  
570696776537202264854470158588016207584749226572260020855844665  
214583988939443709265918003113882464681570826301005948587040031  
864803421948972782906410450726368813137398552561173220402450912  
277002269411275736272804957381089675040183698683684507257993647  
290607629969413804756548237289971803268024744206292691248590521  
810044598421505911202494413417285314781058036033710773091828693  
1471017111168391658172688941975871658215212822951848847 ...

# First binary digits of $\sqrt{2}$

<http://wims.unice.fr/wims/wims.cgi>

1.011010100000100111100110011001111111001110111100110010010000  
10001011001011111011000100110110011011101010100101010111110100  
11111000111010110111101100000101110101000100100111011101010000  
10011001110110100010111101011001000010110000011001100111001100  
10001010101001010111111001000001100000100001110101011100010100  
0101100001110101000101100011111110011011111101110010000011110  
11011001110010000111101110100101010000101111001000011100111000  
11110110100101001111000000001001000011100110110001111011111101  
00010011101101000110100100010000000101110100001110100001010101  
11100011111010011100101001100000101100111000110000000010001101  
11100001100110111101111001010101100011011110010010001000101101  
00010000100010110001010010001100000101010111100011100100010111  
10111110001001110001100111100011011010101101010001010001110001  
01110110111111010011101110011001011001010100110001101000011001  
10001111100111100100001001101111101010010111100010010000011111  
00000110110111001011000001011101110101010100100101000001000100  
110010000010000001100101001001010100000010011100101001010 ...

# Computation of decimals of $\sqrt{2}$

1 542 decimals computed by hand by Horace Uhler in 1951

14 000 decimals computed in 1967

1 000 000 decimals in 1971

$137 \cdot 10^9$  decimals computed by Yasumasa Kanada and Daisuke Takahashi in 1997 with Hitachi SR2201 in 7 hours and 31 minutes.

- Motivation : computation of  $\pi$ .

# Square root of 2 on the web

The first decimal digits of  $\sqrt{2}$  are available on the web

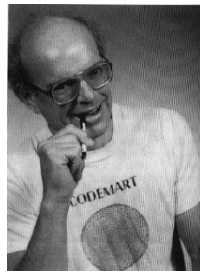
1, 4, 1, 4, 2, 1, 3, 5, 6, 2, 3, 7, 3, 0, 9, 5, 0, 4, 8, 8, 0, 1,  
6, 8, 8, 7, 2, 4, 2, 0, 9, 6, 9, 8, 0, 7, 8, 5, 6, 9, 6, 7, 1, 8, ...

<http://oeis.org/A002193>

The On-Line Encyclopedia of  
Integer Sequences

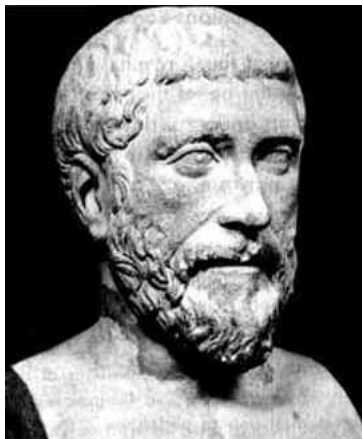
<http://oeis.org/>

Neil J. A. Sloane

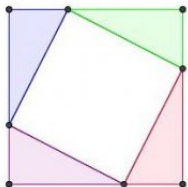
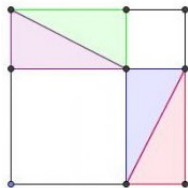




# Pythagoras of Samos $\sim$ 569 BC – $\sim$ 475 BC



$$a^2 + b^2 = c^2 = (a + b)^2 - 2ab.$$



# Irrationality in Greek antiquity



Platon, La République :  
*incommensurable lines,*  
*irrational diagonals.*

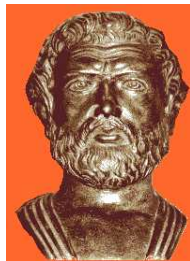
Theodorus of Cyrene  
(about 370 BC.)  
irrationality of  $\sqrt{3}, \dots, \sqrt{17}$ .

**Theetetes** : if an integer  $n > 0$  is the square of a rational number, then it is the square of an integer.

# Irrationality of $\sqrt{2}$



Pythagoreas school



Hippasus of Metapontum (around 500 BC).

Sulba Sutras, Vedic civilization in India, ~800-500 BC.

# Émile Borel : 1950



*The sequence of decimal digits of  $\sqrt{2}$  should behave like a random sequence, each digit should be occurring with the same frequency  $1/10$ , each sequence of 2 digits occurring with the same frequency  $1/100$  ...*

# Émile Borel (1871–1956)

- ▶ *Les probabilités dénombrables et leurs applications arithmétiques,*

Palermo Rend. **27**, 247-271 (1909).

Jahrbuch Database

JFM 40.0283.01

<http://www.emis.de/MATH/JFM/JFM.html>

- ▶ *Sur les chiffres décimaux de  $\sqrt{2}$  et divers problèmes de probabilités en chaînes,*

C. R. Acad. Sci., Paris **230**, 591-593 (1950).

Zbl 0035.08302

# Complexity of the $b$ -ary expansion of an irrational algebraic real number

Let  $b \geq 2$  be an integer.

- **É. Borel** (1909 and 1950) : *the  $b$ -ary expansion of an algebraic irrational number should satisfy some of the laws shared by almost all numbers (with respect to **Lebesgue's measure**)*.
- **Remark** : no number satisfies **all** the laws which are shared by all numbers outside a set of measure zero, because the intersection of all these sets of full measure is empty !

$$\bigcap_{x \in \mathbf{R}} \mathbf{R} \setminus \{x\} = \emptyset.$$

- More precise statements by **B. Adamczewski** and **Y. Bugeaud**.

# Conjecture of Émile Borel

**Conjecture** (É. Borel). Let  $x$  be an irrational algebraic real number,  $b \geq 3$  a positive integer and  $a$  an integer in the range  $0 \leq a \leq b - 1$ . Then the digit  $a$  occurs at least once in the  $b$ -ary expansion of  $x$ .

**Corollary.** Each given sequence of digits should occur infinitely often in the  $b$ -ary expansion of any real irrational algebraic number.

(consider powers of  $b$ ).

- An irrational number with a *regular* expansion in some basis  $b$  should be transcendental.

# The state of the art

There is no explicitly known example of a triple  $(b, a, x)$ , where  $b \geq 3$  is an integer,  $a$  is a digit in  $\{0, \dots, b-1\}$  and  $x$  is an algebraic irrational number, for which one can claim that the digit  $a$  occurs infinitely often in the  $b$ -ary expansion of  $x$ .

A stronger conjecture, also due to Borel, is that algebraic irrational real numbers are *normal*: each sequence of  $n$  digits in basis  $b$  should occur with the frequency  $1/b^n$ , for all  $b$  and all  $n$ .



# What is known on the decimal expansion of $\sqrt{2}$ ?

The sequence of digits (in any basis) of  $\sqrt{2}$  is not ultimately periodic

Among the decimal digits

$$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\},$$

at least two of them occur infinitely often. Almost nothing else is known.

# Complexity of the expansion in basis $b$ of a real irrational algebraic number



**Theorem** (B. Adamczewski, Y. Bugeaud 2005 ; conjecture of A. Cobham 1968).

*If the sequence of digits of a real number  $x$  is produced by a finite automaton, then  $x$  is either rational or else transcendental.*

## §2 Irrationality of transcendental numbers

- The number  $e$
- The number  $\pi$
- Open problems

# Introductio in analysin infinitorum



$e$  is irrational.

Leonhard Euler (1737)

(1707 – 1783)

Introductio in analysin infinitorum

Continued fraction of  $e$  :

$$e = 2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{4 + \frac{1}{\ddots}}}}}}}$$

# Joseph Fourier

Fourier (1815) : proof by means of the series expansion

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{N!} + r_N$$

with  $r_N > 0$  and  $N!r_N \rightarrow 0$  as  $N \rightarrow +\infty$ .



Course of analysis at the École Polytechnique Paris, 1815.

# Variant of Fourier's proof : $e^{-1}$ is irrational

F. Beukers : alternating series

For odd  $N$ ,

$$1 - \frac{1}{1!} + \frac{1}{2!} - \dots - \frac{1}{N!} < e^{-1} < 1 - \frac{1}{1!} + \frac{1}{2!} - \dots + \frac{1}{(N+1)!}$$

$$\frac{a_N}{N!} < e^{-1} < \frac{a_N}{N!} + \frac{1}{(N+1)!}, \quad a_N \in \mathbf{Z}$$

$$a_N < N!e^{-1} < a_N + 1.$$

Hence  $N!e^{-1}$  is not an integer.

# Irrationality of $\pi$

$\bar{\text{A}}\text{ryabha\text{ṭa}}$ , born 476 AD :  $\pi \sim 3.1416$ .

$\text{N}\bar{\text{i}}\text{lakaṅ\text{ṭha}} \text{ Somayā\text{j}\bar{\text{i}}}$ , born 1444 AD : *Why then has an approximate value been mentioned here leaving behind the actual value? Because it (exact value) cannot be expressed.*

$\text{K. Ramasubramanian}$ , *The Notion of Proof in Indian Science*,  
13th World Sanskrit Conference, 2006.

# Irrationality of $\pi$

Johann Heinrich Lambert (1728 – 1777)

*Mémoire sur quelques propriétés remarquables des quantités transcendentes circulaires et logarithmiques,*

Mémoires de l'Académie des Sciences de Berlin, **17** (1761), p. 265-322 ;  
lu en 1767 ; Math. Werke, t. II.



$\tan(v)$  is irrational when  $v \neq 0$  is rational.

As a consequence,  $\pi$  is irrational, since  $\tan(\pi/4) = 1$ .



# Lambert and Frederick II, King of Prussia



- Que savez vous, Lambert ?
- Tout, Sire.
- Et de qui le tenez-vous ?
- De moi-même !



# Known and unknown transcendence results

Known :

$$e, \pi, \log 2, e^{\sqrt{2}}, e^{\pi}, 2^{\sqrt{2}}, \Gamma(1/4).$$

Not known :

$$e + \pi, e\pi, \log \pi, \pi^e, \Gamma(1/5), \zeta(3), \text{Euler constant}$$

Why is  $e^{\pi}$  known to be transcendental while  $\pi^e$  is not known to be irrational?

Answer :  $e^{\pi} = (-1)^{-i}$ .

# Catalan's constant

Is Catalan's constant

$$\sum_{n \geq 1} \frac{(-1)^n}{(2n+1)^2}$$

= 0.915 965 594 177 219 015 0 ...

an irrational number?



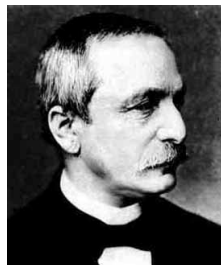
# Catalan's constant, Dirichlet and Kronecker

Catalan's constant is the value at  $s = 2$  of the Dirichlet  $L$ -function  $L(s, \chi_{-4})$  associated with the Kronecker character

$$\chi_{-4}(n) = \left(\frac{n}{4}\right) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \equiv 1 \pmod{4}, \\ -1 & \text{if } n \equiv -1 \pmod{4}. \end{cases}$$



Johann Peter Gustav Lejeune Dirichlet  
1805 – 1859



Leopold Kronecker  
1823 – 1891

# Catalan's constant, Dedekind and Riemann

The Dirichlet  $L$ -function  $L(s, \chi_{-4})$  associated with the Kronecker character  $\chi_{-4}$  is the quotient of the Dedekind zeta function of  $\mathbf{Q}(i)$  and the Riemann zeta function :

$$\zeta_{\mathbf{Q}(i)}(s) = L(s, \chi_{-4})\zeta(s)$$



Julius Wilhelm Richard  
Dedekind  
1831 – 1916



Georg Friedrich Bernhard  
Riemann  
1826 – 1866

# Riemann zeta function



The function

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$$

was studied by Euler (1707– 1783)

for integer values of  $s$

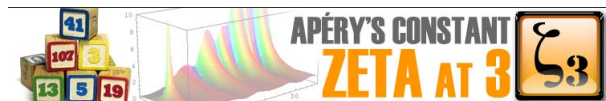
and by Riemann (1859) for complex values of  $s$ .

Euler : for any even integer value of  $s \geq 2$ , the number  $\zeta(s)$  is a rational multiple of  $\pi^s$ .

Examples :  $\zeta(2) = \pi^2/6$ ,  $\zeta(4) = \pi^4/90$ ,  $\zeta(6) = \pi^6/945$ ,  
 $\zeta(8) = \pi^8/9450 \dots$

Coefficients : Bernoulli numbers.

# Riemann zeta function



The number

$$\zeta(3) = \sum_{n \geq 1} \frac{1}{n^3} = 1,202\,056\,903\,159\,594\,285\,399\,738\,161\,511 \dots$$

is irrational (*Apéry 1978*).

Recall that  $\zeta(s)/\pi^s$  is rational for any even value of  $s \geq 2$ .

**Open question** : Is the number  $\zeta(3)/\pi^3$  irrational ?

# Riemann zeta function

Is the number

$$\zeta(5) = \sum_{n \geq 1} \frac{1}{n^5} = 1.036\,927\,755\,143\,369\,926\,331\,365\,486\,457 \dots$$

irrational ?

*T. Rivoal* (2000) : infinitely many  $\zeta(2n + 1)$  are irrational.



# Infinitely many odd zeta values are irrational

Tanguy Rivoal (2000)

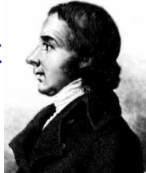
Let  $\epsilon > 0$ . For any sufficiently large odd integer  $a$ , the dimension of the  $\mathbf{Q}$ -vector space spanned by the numbers  $1, \zeta(3), \zeta(5), \dots, \zeta(a)$  is at least

$$\frac{1 - \epsilon}{1 + \log 2} \log a.$$





# Euler–Mascheroni constant



Euler's Constant is

Lorenzo Mascheroni  
(1750 – 1800)

$$\gamma = \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n \right)$$

$$= 0.577\ 215\ 664\ 901\ 532\ 860\ 606\ 512\ 090\ 082 \dots$$

Is it a rational number?

$$\gamma = \sum_{k=1}^{\infty} \left( \frac{1}{k} - \log \left( 1 + \frac{1}{k} \right) \right) = \int_1^{\infty} \left( \frac{1}{[x]} - \frac{1}{x} \right) dx$$

$$= - \int_0^1 \int_0^1 \frac{(1-x) dx dy}{(1-xy) \log(xy)}$$

# Euler's constant

Recent work by *J. Sondow* inspired by the work of *F. Beukers* on *Apéry's* proof.



F. Beukers



Jonathan Sondow

<http://home.earthlink.net/~jsondow/>



$$\gamma = \int_0^{\infty} \sum_{k=2}^{\infty} \frac{1}{k^2 \binom{t+k}{k}} dt$$

$$\gamma = \lim_{s \rightarrow 1^+} \sum_{n=1}^{\infty} \left( \frac{1}{n^s} - \frac{1}{s^n} \right)$$

$$\gamma = \int_1^{\infty} \frac{1}{2t(t+1)} F\left( \begin{matrix} 1, & 2, & 2 \\ 3, & t+2 \end{matrix} \right) dt.$$

# Euler Gamma function

Is the number

$$\Gamma(1/5) = 4.590\ 843\ 711\ 998\ 803\ 053\ 204\ 758\ 275\ 929\ 152\ \dots$$

irrational?

$$\Gamma(z) = e^{-\gamma z} z^{-1} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n} = \int_0^{\infty} e^{-t} t^z \cdot \frac{dt}{t}$$

Here is the set of rational values for  $z \in (0, 1)$  for which the answer is known (and, for these arguments, the Gamma value is a transcendental number) :

$$r \in \left\{ \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6} \right\} \quad (\text{mod } 1).$$

## Georg Cantor (1845 - 1918)



The set of algebraic numbers is countable, not the set of real (or complex) numbers.

Cantor (1874 and 1891).

# Henri Léon Lebesgue (1875 – 1941)

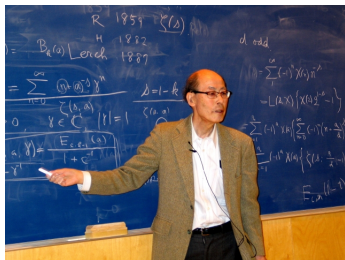
Almost all numbers for  
Lebesgue measure are  
transcendental numbers.



# Most numbers are transcendental

**Meta conjecture** : *any number given by some kind of limit, which is not obviously rational (resp. algebraic), is irrational (resp. transcendental).*

Goro Shimura





# Special values of hypergeometric series

Jürgen Wolfart



Frits Beukers



# Sum of values of a rational function

Work by S.D. Adhikari, N. Saradha, T.N. Shorey and R. Tijdeman (2001),

Let  $P$  and  $Q$  be non-zero polynomials having rational coefficients and  $\deg Q \geq 2 + \deg P$ . Consider

$$\sum_{\substack{n \geq 0 \\ Q(n) \neq 0}} \frac{P(n)}{Q(n)}.$$

Robert Tijdeman



Sukumar Das Adhikari



N. Saradha



# Telescoping series

## Examples

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1, \quad \sum_{n=0}^{\infty} \frac{1}{n^2 - 1} = \frac{3}{4},$$

$$\sum_{n=0}^{\infty} \left( \frac{1}{4n+1} - \frac{3}{4n+2} + \frac{1}{4n+3} + \frac{1}{4n+4} \right) = 0$$

$$\sum_{n=0}^{\infty} \left( \frac{1}{5n+2} - \frac{3}{5n+7} + \frac{1}{5n-3} \right) = \frac{5}{6}$$

# Transcendental values

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+2)} = \log 2,$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)(2n+1)(4n+1)} = \frac{\pi}{3}$$

are transcendental.

# Transcendental values

$$\sum_{n=0}^{\infty} \frac{1}{(6n+1)(6n+2)(6n+3)(6n+4)(6n+5)(6n+6)}$$
$$= \frac{1}{4320} (192 \log 2 - 81 \log 3 - 7\pi\sqrt{3})$$

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1} = \frac{1}{2} + \frac{\pi}{2} \cdot \frac{e^{\pi} + e^{-\pi}}{e^{\pi} - e^{-\pi}} = 2.076\,674\,047\,4\dots$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1} = \frac{2\pi}{e^{\pi} - e^{-\pi}} = 0.272\,029\,054\,982\dots$$

# Leonardo Pisano (Fibonacci)

The Fibonacci sequence

$(F_n)_{n \geq 0}$  :

0, 1, 1, 2, 3, 5, 8, 13, 21,

34, 55, 89, 144, 233...

is defined by

$$F_0 = 0, F_1 = 1,$$

$$F_n = F_{n-1} + F_{n-2} \quad (n \geq 2).$$

Leonardo Pisano (Fibonacci)

(1170–1250)



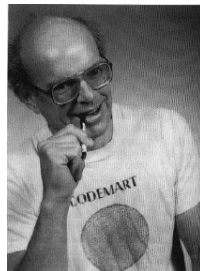
# Encyclopedia of integer sequences (again)

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597,  
2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418,  
317811, 514229, 832040, 1346269, 2178309, 3524578, 5702887, 9227465, ...

The **Fibonacci** sequence is  
available online

**The On-Line Encyclopedia  
of Integer Sequences**

Neil J. A. Sloane



<http://oeis.org/A000045>

# Series involving Fibonacci numbers

The number

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2}} = 1$$

is rational, while

$$\sum_{n=0}^{\infty} \frac{1}{F_{2^n}} = \frac{7 - \sqrt{5}}{2}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+1}} = \frac{1 - \sqrt{5}}{2}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1} + 1} = \frac{\sqrt{5}}{2}$$

are irrational algebraic numbers.



# Series involving Fibonacci numbers

The numbers

$$\sum_{n=1}^{\infty} \frac{1}{F_n^2}, \quad \sum_{n=1}^{\infty} \frac{1}{F_n^4}, \quad \sum_{n=1}^{\infty} \frac{1}{F_n^6},$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n^2}, \quad \sum_{n=1}^{\infty} \frac{n}{F_{2n}},$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1} + F_{2n+1}}, \quad \sum_{n=1}^{\infty} \frac{1}{F_{2n+1}}$$

are all transcendental

# Series involving Fibonacci numbers

Each of the numbers

$$\sum_{n=1}^{\infty} \frac{1}{F_n}, \quad \sum_{n=1}^{\infty} \frac{1}{F_n + F_{n+2}}$$

$$\sum_{n \geq 1} \frac{1}{F_1 F_2 \cdots F_n}$$

is irrational, but it is not known whether they are algebraic or transcendental.

The first challenge here is to formulate a conjectural statement which would give a satisfactory description of the situation.

# The Fibonacci zeta function

For  $\Re(s) > 0$ ,

$$\zeta_F(s) = \sum_{n \geq 1} \frac{1}{F_n^s}$$

$\zeta_F(2)$ ,  $\zeta_F(4)$ ,  $\zeta_F(6)$  are algebraically independent.

Iekata Shiokawa, Carsten Elsner and Shun Shimomura (2006)



Iekata Shiokawa

## §3 Transcendental numbers

- Liouville (1844)
- Hermite (1873)
- Lindemann (1882)
- Hilbert's Problem 7th (1900)
- Gel'fond–Schneider (1934)
- Baker (1968)
- Nesterenko (1995)

# Existence of transcendental numbers (1844)

*J. Liouville (1809 - 1882)*

gave the first examples of  
transcendental numbers.

For instance

$$\sum_{n \geq 1} \frac{1}{10^{n!}} = 0.110\,001\,000\,000\,0 \dots$$

is a transcendental number.



# Charles Hermite and Ferdinand Lindemann



*Hermite (1873) :*  
Transcendence of  $e$   
 $e = 2.718\ 281\ 828\ 4\dots$



*Lindemann (1882) :*  
Transcendence of  $\pi$   
 $\pi = 3.141\ 592\ 653\ 5\dots$

# Hermite–Lindemann Theorem

*For any non-zero complex number  $z$ , one at least of the two numbers  $z$  and  $e^z$  is transcendental.*

*Corollaries* : Transcendence of  $\log \alpha$  and of  $e^\beta$  for  $\alpha$  and  $\beta$  non-zero algebraic complex numbers, provided  $\log \alpha \neq 0$ .

# Transcendental functions

A complex function is called **transcendental** if it is transcendental over the field  $\mathbf{C}(z)$ , which means that the functions  $z$  and  $f(z)$  are algebraically independent : if  $P \in \mathbf{C}[X, Y]$  is a non-zero polynomial, then the function  $P(z, f(z))$  is not 0.

**Exercise.** *An entire function (analytic in  $\mathbf{C}$ ) is transcendental if and only if it is not a polynomial.*

**Example.** *The transcendental entire function  $e^z$  takes an algebraic value at an algebraic argument  $z$  only for  $z = 0$ .*



# Weierstrass question

*Is it true that a transcendental entire function  $f$  takes usually transcendental values at algebraic arguments?*



**Examples :** for  $f(z) = e^z$ , there is a single exceptional point  $\alpha$  algebraic with  $e^\alpha$  also algebraic, namely  $\alpha = 0$ .

For  $f(z) = e^{P(z)}$  where  $P \in \mathbf{Z}[z]$  is a non-constant polynomial, there are finitely many exceptional points  $\alpha$ , namely the roots of  $P$ .

The exceptional set of  $e^z + e^{1+z}$  is empty (Lindemann–Weierstrass).

The exceptional set of functions like  $2^z$  or  $e^{i\pi z}$  is  $\mathbf{Q}$ , (Gel'fond and Schneider).

# Exceptional sets

Answers by Weierstrass (letter to Strauss in 1886), Strauss, Stäckel, Faber, van der Poorten, Gramain. . .

*If  $S$  is a countable subset of  $\mathbf{C}$  and  $T$  is a dense subset of  $\mathbf{C}$ , there exist transcendental entire functions  $f$  mapping  $S$  into  $T$ , as well as all its derivatives.*

*Any set of algebraic numbers is the exceptional set of some transcendental entire function.*

Also multiplicities can be included.

*van der Poorten : there are transcendental entire functions  $f$  such that  $D^k f(\alpha) \in \mathbf{Q}(\alpha)$  for all  $k \geq 0$  and all algebraic  $\alpha$ .*

# Integer valued entire functions

An **integer valued entire function** is a function  $f$ , which is analytic in  $\mathbf{C}$ , and maps  $\mathbf{N}$  into  $\mathbf{Z}$ .

Example :  $2^z$  is an integer valued entire function, not a polynomial.

**Question** : Are there integer valued entire function growing slower than  $2^z$  without being a polynomial?

Let  $f$  be a transcendental entire function in  $\mathbf{C}$ . For  $R > 0$  set

$$|f|_R = \sup_{|z|=R} |f(z)|.$$

# Integer valued entire functions

*G. Pólya* (1914) :  
if  $f$  is not a polynomial  
and  $f(n) \in \mathbf{Z}$  for  $n \in \mathbf{Z}_{\geq 0}$ , then  
$$\limsup_{R \rightarrow \infty} 2^{-R} |f|_R \geq 1.$$



Further works on this topic by G.H. Hardy, G. Pólya, D. Sato,  
E.G. Straus, A. Selberg, Ch. Pisot, F. Carlson, F. Gross, . . .

# Integer valued entire function on $\mathbf{Z}[i]$

*A.O. Gel'fond (1929)* : growth of entire functions mapping the Gaussian integers into themselves.

*Newton* interpolation series at the points in  $\mathbf{Z}[i]$ .

*An entire function  $f$  which is not a polynomial and satisfies  $f(a + ib) \in \mathbf{Z}[i]$  for all  $a + ib \in \mathbf{Z}[i]$  satisfies*

$$\limsup_{R \rightarrow \infty} \frac{1}{R^2} \log |f|_R \geq \delta.$$

*F. Gramain (1981)* :  $\delta = \pi/(2e) = 0,577\,863\,674\,8\dots$

This is best possible : *D.W. Masser (1980)*.

# Transcendence of $e^\pi$

*A.O. Gel'fond (1929).*



If

$$e^\pi = 23.140\,692\,632\,779\,269\,005\,729\,086\,367 \dots$$

is rational, then the function  $e^{\pi z}$  takes values in  $\mathbf{Q}(i)$  when the argument  $z$  is in  $\mathbf{Z}[i]$ .

Expand  $e^{\pi z}$  into an interpolation series at the Gaussian integers.

# Hilbert's Problems

August 8, 1900



David Hilbert (1862 - 1943)

Second International Congress  
of Mathematicians in Paris.

Twin primes,

Goldbach's Conjecture,

Riemann Hypothesis

Transcendence of  $e^\pi$  and  $2^{\sqrt{2}}$

## A.O. Gel'fond and Th. Schneider

Solution of Hilbert's seventh problem (1934) : *Transcendence of  $\alpha^\beta$  and of  $(\log \alpha_1)/(\log \alpha_2)$  for algebraic  $\alpha, \beta, \alpha_1$  and  $\alpha_2$ .*





# Transcendence of $\alpha^\beta$ and $\log \alpha_1 / \log \alpha_2$ : examples

The following numbers are transcendental :

$$2^{\sqrt{2}} = 2.665\,144\,142\,6\dots$$

$$\frac{\log 2}{\log 3} = 0.630\,929\,753\,5\dots$$

$$e^\pi = 23.140\,692\,632\,7\dots \quad (e^\pi = (-1)^{-i})$$

$$e^{\pi\sqrt{163}} = 262\,537\,412\,640\,768\,743.999\,999\,999\,999\,25\dots$$

$$e^{\pi} = (-1)^{-i}$$

*Example* : Transcendence of the number

$$e^{\pi\sqrt{163}} = 262\,537\,412\,640\,768\,743.999\,999\,999\,999\,2\dots$$

*Remark.* For

$$\tau = \frac{1 + i\sqrt{163}}{2}, \quad q = e^{2i\pi\tau} = -e^{-\pi\sqrt{163}}$$

we have  $j(\tau) = -640\,320^3$  and

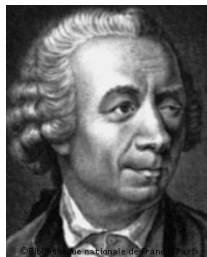
$$\left| j(\tau) - \frac{1}{q} - 744 \right| < 10^{-12}.$$

# Beta values : Th. Schneider 1948

Euler Gamma and Beta functions

$$B(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1} dx.$$

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^z \cdot \frac{dt}{t}$$



$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$



# Algebraic independence : A.O. Gel'fond 1948



The two numbers  $2^{\sqrt[3]{2}}$  and  $2^{\sqrt[3]{4}}$  are algebraically independent.

*More generally*, if  $\alpha$  is an algebraic number,  $\alpha \neq 0$ ,  $\alpha \neq 1$  and if  $\beta$  is an algebraic number of degree  $d \geq 3$ , then two at least of the numbers

$$\alpha^\beta, \alpha^{\beta^2}, \dots, \alpha^{\beta^{d-1}}$$

are algebraically independent.

# Alan Baker 1968

Transcendence of numbers  
like

$$\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$$

or

$$e^{\beta_0} \alpha_1^{\beta_1} \cdots \alpha_n^{\beta_n}$$

for algebraic  $\alpha_i$ 's and  $\beta_j$ 's.

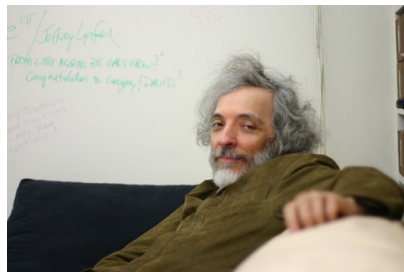


*Example* (Siegel) :

$$\int_0^1 \frac{dx}{1+x^3} = \frac{1}{3} \left( \log 2 + \frac{\pi}{\sqrt{3}} \right) = 0.835\,648\,848 \dots$$

is transcendental.

# Gregory V. Chudnovsky



G.V. Chudnovsky (1976)

Algebraic independence of the numbers  $\pi$  and  $\Gamma(1/4)$ .

Also : algebraic independence of the numbers  $\pi$  and  $\Gamma(1/3)$ .

Corollaries : *Transcendence of*  $\Gamma(1/4) = 3.625\ 609\ 908\ 2\dots$   
and  $\Gamma(1/3) = 2.678\ 938\ 534\ 7\dots$

# Yuri V. Nesterenko



Yu.V.Nesterenko (1996)  
Algebraic independence of  
 $\Gamma(1/4)$ ,  $\pi$  and  $e^\pi$ .  
Also : Algebraic  
independence of  
 $\Gamma(1/3)$ ,  $\pi$  and  $e^{\pi\sqrt{3}}$ .

*Corollary : The numbers  $\pi = 3.141\,592\,6535\dots$  and  $e^\pi = 23.140\,692\,6327\dots$  are algebraically independent.*

Transcendence of values of Dirichlet's  $L$ -functions :  
Sanoli Gun, Ram Murty and Purusottam Rath (2009).

# Weierstraß sigma function

Let  $\Omega = \mathbf{Z}\omega_1 + \mathbf{Z}\omega_2$  be a lattice in  $\mathbf{C}$ . The canonical product attached to  $\Omega$  is the *Weierstraß sigma function*

$$\sigma(z) = \sigma_{\Omega}(z) = z \prod_{\omega \in \Omega \setminus \{0\}} \left(1 - \frac{z}{\omega}\right) e^{(z/\omega) + (z^2/2\omega^2)}.$$

The number

$$\sigma_{\mathbf{Z}[i]}(1/2) = 2^{5/4} \pi^{1/2} e^{\pi/8} \Gamma(1/4)^{-2}$$

is transcendental.



## §4 : Conjectures

Borel 1909, 1950

Schanuel 1964

Grothendieck 1960's

Rohrlich and Lang 1970's

André 1990's

Kontsevich and Zagier 2001.

# Periods : Maxime Kontsevich and Don Zagier



Periods,  
*Mathematics  
unlimited—2001  
and beyond*,  
Springer 2001,  
771–808.



A *period* is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients, over domains in  $\mathbf{R}^n$  given by polynomial inequalities with rational coefficients.

# The number $\pi$

Basic example of a *period* :

$$e^{z+2i\pi} = e^z$$

$$2i\pi = \int_{|z|=1} \frac{dz}{z}$$

$$\begin{aligned}\pi &= \iint_{x^2+y^2 \leq 1} dx dy = 2 \int_{-1}^1 \sqrt{1-x^2} dx \\ &= \int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} = \int_{-\infty}^{\infty} \frac{dx}{1-x^2}.\end{aligned}$$

## Further examples of periods

$$\sqrt{2} = \int_{2x^2 \leq 1} dx$$

and all algebraic numbers.

$$\log 2 = \int_{1 < x < 2} \frac{dx}{x}$$

and all logarithms of algebraic numbers.

$$\pi = \int_{x^2 + y^2 \leq 1} dx dy,$$

A product of periods is a period (subalgebra of  $\mathbf{C}$ ), but  $1/\pi$  is expected not to be a period.

# Relations among periods

## 1 Additivity

(in the integrand and in the domain of integration)

$$\int_a^b (f(x) + g(x)) dx = \int_a^b f(x) dx + \int_a^b g(x) dx,$$

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

## 2 Change of variables :

if  $y = f(x)$  is an invertible change of variables, then

$$\int_{f(a)}^{f(b)} F(y) dy = \int_a^b F(f(x)) f'(x) dx.$$

## Relations among periods (continued)



### 3 Newton–Leibniz–Stokes Formula

$$\int_a^b f'(x) dx = f(b) - f(a).$$

# Conjecture of Kontsevich and Zagier



A widely-held belief, based on a judicious combination of experience, analogy, and wishful thinking, is the following



**Conjecture (Kontsevich–Zagier).** *If a period has two integral representations, then one can pass from one formula to another by using only rules  $\boxed{1}$ ,  $\boxed{2}$ ,  $\boxed{3}$  in which all functions and domains of integration are algebraic with algebraic coefficients.*

# Conjecture of Kontsevich and Zagier (continued)

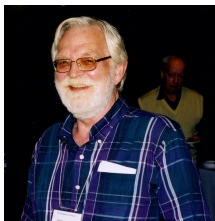
*In other words, we do not expect any miraculous coincidence of two integrals of algebraic functions which will not be possible to prove using three simple rules.*

*This conjecture, which is similar in spirit to the Hodge conjecture, is one of the central conjectures about algebraic independence and transcendental numbers, and is related to many of the results and ideas of modern arithmetic algebraic geometry and the theory of motives.*

**Advice** : *if you wish to prove a number is transcendental, first prove it is a period.*



# Conjectures by S. Schanuel and A. Grothendieck



- **Schanuel** : if  $x_1, \dots, x_n$  are  $\mathbf{Q}$ -linearly independent complex numbers, then *n* at least of the  $2n$  numbers  $x_1, \dots, x_n, e^{x_1}, \dots, e^{x_n}$  are algebraically independent.
- **Periods conjecture by Grothendieck** : Dimension of the Mumford–Tate group of a smooth projective variety.

# Motives



Y. André : generalization of Grothendieck's conjecture to motives.



Case of 1–motives :  
Elliptico-Toric Conjecture of  
C. Bertolin.

July 15, 2013

Institute of Mathematics of  
University of the Philippines Diliman

# Transcendental Number Theory: recent results and open problems.

*Michel Waldschmidt*

Institut de Mathématiques de Jussieu  
<http://www.math.jussieu.fr/~miw/>