

WIEN, ESI

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SYMPOSIUM ON DIOPHANTINE PROBLEMS
IN HONOUR OF WOLFGANG SCHMIDT

Dependence of logarithms on commutative algebraic groups

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Algebraic independence of logarithms of algebraic numbers

Conjecture *Let $\alpha_1, \dots, \alpha_n$ be non zero algebraic numbers. For $1 \leq j \leq n$ let $\lambda_j \in \mathbf{C}$ satisfy $e^{\lambda_j} = \alpha_j$. Assume $\lambda_1, \dots, \lambda_n$ are linearly independent over \mathbf{Q} . Then $\lambda_1, \dots, \lambda_n$ are algebraically independent.*

Write $\lambda_j = \log \alpha_j$.

If $\log \alpha_1, \dots, \log \alpha_n$ are \mathbf{Q} -linearly independent then they are algebraically independent.

Denote by \mathcal{L} the \mathbf{Q} vector space of logarithms of algebraic numbers:

$$\mathcal{L} = \{\lambda \in \mathbf{C} ; e^\lambda \in \overline{\mathbf{Q}}\} = \{\log \alpha ; \alpha \in \overline{\mathbf{Q}}^\times\} = \exp^{-1}(\overline{\mathbf{Q}}^\times).$$

The conjecture on algebraic independence of logarithms of algebraic numbers can be phrased:

The injection of \mathcal{L} into \mathbf{C} extends into an injection of the symmetric algebra $\text{Sym}_{\mathbf{Q}}(\mathcal{L})$ on \mathcal{L} into \mathbf{C} .

Four Exponentials Conjecture For $i = 1, 2$ and $j = 1, 2$, let α_{ij} be a non zero algebraic number and λ_{ij} a complex number satisfying $e^{\lambda_{ij}} = \alpha_{ij}$. Assume $\lambda_{11}, \lambda_{12}$ are linearly independent over \mathbf{Q} and also $\lambda_{11}, \lambda_{21}$ are linearly independent over \mathbf{Q} . Then

$$\lambda_{11}\lambda_{22} \neq \lambda_{12}\lambda_{21}.$$

Four Exponentials Conjecture For $i = 1, 2$ and $j = 1, 2$, let α_{ij} be a non zero algebraic number and λ_{ij} a complex number satisfying $e^{\lambda_{ij}} = \alpha_{ij}$. Assume $\lambda_{11}, \lambda_{12}$ are linearly independent over \mathbf{Q} and also $\lambda_{11}, \lambda_{21}$ are linearly independent over \mathbf{Q} . Then

$$\lambda_{11}\lambda_{22} \neq \lambda_{12}\lambda_{21}.$$

Notice:

$$\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21} = \det \begin{vmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{12} & \lambda_{22} \end{vmatrix}.$$

Four Exponentials Conjecture (again) *Let x_1, x_2 be two \mathbb{Q} -linearly independent complex numbers and y_1, y_2 also two \mathbb{Q} -linearly independent complex numbers. Then one at least of the four numbers*

$$e^{x_1 y_1}, e^{x_1 y_2}, e^{x_2 y_1}, e^{x_2 y_2}$$

is transcendental.

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$$e^{x_1 y_1}, e^{x_1 y_2}, e^{x_2 y_1}, e^{x_2 y_2}$$

is transcendental.

Hint: Set

$$x_i y_j = \lambda_{ij} \quad (i = 1, 2; j = 1, 2).$$

A rank one matrix is a matrix of the form

$$\begin{pmatrix} x_1 y_1 & x_1 y_2 \\ x_2 y_1 & x_2 y_2 \end{pmatrix}.$$

Sharp Four Exponentials Conjecture. *If x_1, x_2 are two complex numbers which are \mathbb{Q} -linearly independent, if y_1, y_2 , are two complex numbers which are \mathbb{Q} -linearly independent and if $\beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}$ are four algebraic numbers such that the four numbers*

$$e^{x_1 y_1 - \beta_{11}}, e^{x_1 y_2 - \beta_{12}}, e^{x_2 y_1 - \beta_{21}}, e^{x_2 y_2 - \beta_{22}}$$

are algebraic, then $x_i y_j = \beta_{ij}$ for $i = 1, 2$ and $j = 1, 2$.

Sharp Four Exponentials Conjecture. *If x_1, x_2 are two complex numbers which are \mathbb{Q} -linearly independent, if y_1, y_2 , are two complex numbers which are \mathbb{Q} -linearly independent and if $\beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}$ are four algebraic numbers such that the four numbers*

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are algebraic, then $x_i y_j = \beta_{ij}$ for $i = 1, 2$ and $j = 1, 2$.

If $x_i y_j = \lambda_{ij} + \beta_{ij}$ then

$$x_1 x_2 y_1 y_2 = (\lambda_{11} + \beta_{11})(\lambda_{22} + \beta_{22}) = (\lambda_{12} + \beta_{12})(\lambda_{21} + \beta_{21}).$$

Denote by $\tilde{\mathcal{L}}$ the $\overline{\mathbf{Q}}$ -vector space spanned by 1 and \mathcal{L} (linear combinations of logarithms of algebraic numbers with algebraic coefficients):

$$\tilde{\mathcal{L}} = \left\{ \beta_0 + \sum_{h=1}^{\ell} \beta_h \log \alpha_h ; \ell \geq 0, \alpha\text{'s in } \overline{\mathbf{Q}}^{\times}, \beta\text{'s in } \overline{\mathbf{Q}} \right\}$$

Strong Four Exponentials Conjecture. *If x_1, x_2 are $\overline{\mathbf{Q}}$ -linearly independent and if y_1, y_2 , are $\overline{\mathbf{Q}}$ -linearly independent, then one at least of the four numbers*

$$x_1 y_1, x_1 y_2, x_2 y_1, x_2 y_2$$

does not belong to $\tilde{\mathcal{L}}$.

Six Exponentials Theorem. *If x_1, x_2 are two complex numbers which are \mathbb{Q} -linearly independent, if y_1, y_2, y_3 are three complex numbers which are \mathbb{Q} -linearly independent, then one at least of the six numbers*

$$e^{x_i y_j} \quad (i = 1, 2, j = 1, 2, 3)$$

is transcendental.

Sharp Six Exponentials Theorem. *If x_1, x_2 are two complex numbers which are \mathbf{Q} -linearly independent, if y_1, y_2, y_3 are three complex numbers which are \mathbf{Q} -linearly independent and if β_{ij} are six algebraic numbers such that*

$$e^{x_i y_j - \beta_{ij}} \in \overline{\mathbf{Q}} \quad \text{for } i = 1, 2, j = 1, 2, 3,$$

then $x_i y_j = \beta_{ij}$ for $i = 1, 2$ and $j = 1, 2, 3$.

Strong Six Exponentials Theorem (D. Roy). *If x_1, x_2 are $\overline{\mathbb{Q}}$ -linearly independent and if y_1, y_2, y_3 are $\overline{\mathbb{Q}}$ -linearly independent, then one at least of the six numbers*

$$x_i y_j \quad (i = 1, 2, j = 1, 2, 3)$$

does not belong to $\tilde{\mathcal{L}}$.

Five Exponentials Theorem. *If x_1, x_2 are \mathbb{Q} -linearly independent, y_1, y_2 are \mathbb{Q} -linearly independent and γ is a non zero algebraic number, then one at least of the five numbers*

$$e^{x_1 y_1}, e^{x_1 y_2}, e^{x_2 y_1}, e^{x_2 y_2}, e^{\gamma x_2 / x_1}$$

is transcendental.

This is a consequence of the sharp 6 exponentials Theorem: set $y_3 = \gamma / x_1$ and use Baker's Theorem for checking that y_1, y_2, y_3 are linearly independent over \mathbb{Q} .

Sharp Five Exponentials Conjecture. *If x_1, x_2 are \mathbb{Q} -linearly independent, if y_1, y_2 are \mathbb{Q} -linearly independent and if $\alpha, \beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}, \gamma$ are six algebraic numbers such that*

$$e^{x_1 y_1 - \beta_{11}}, e^{x_1 y_2 - \beta_{12}}, e^{x_2 y_1 - \beta_{21}}, e^{x_2 y_2 - \beta_{22}}, e^{(\gamma x_2 / x_1) - \alpha}$$

are algebraic, then $x_i y_j = \beta_{ij}$ for $i = 1, 2, j = 1, 2$ and also $\gamma x_2 = \alpha x_1$.

Sharp Five Exponentials Conjecture. *If x_1, x_2 are \mathbb{Q} -linearly independent, if y_1, y_2 are \mathbb{Q} -linearly independent and if $\alpha, \beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}, \gamma$ are six algebraic numbers such that*

$$e^{x_1 y_1 - \beta_{11}}, e^{x_1 y_2 - \beta_{12}}, e^{x_2 y_1 - \beta_{21}}, e^{x_2 y_2 - \beta_{22}}, e^{(\gamma x_2 / x_1) - \alpha}$$

are algebraic, then $x_i y_j = \beta_{ij}$ for $i = 1, 2, j = 1, 2$ and also $\gamma x_2 = \alpha x_1$.

Difficult case: when $y_1, y_2, \gamma/x_1$ are \mathbb{Q} -linearly dependent.

Example: $x_1 = y_1 = \gamma = 1$.

Sharp Five Exponentials Conjecture. *If x_1, x_2 are \mathbb{Q} -linearly independent, if y_1, y_2 are \mathbb{Q} -linearly independent and if $\alpha, \beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}, \gamma$ are six algebraic numbers such that*

$$e^{x_1 y_1 - \beta_{11}}, e^{x_1 y_2 - \beta_{12}}, e^{x_2 y_1 - \beta_{21}}, e^{x_2 y_2 - \beta_{22}}, e^{(\gamma x_2 / x_1) - \alpha}$$

are algebraic, then $x_i y_j = \beta_{ij}$ for $i = 1, 2, j = 1, 2$ and also $\gamma x_2 = \alpha x_1$.

Consequence: *Transcendence of the number e^{π^2} .*

Proof. Set $x_1 = y_1 = 1, x_2 = y_2 = i\pi, \gamma = 1, \alpha = 0, \beta_{11} = 1, \beta_{ij} = 0$ for $(i, j) \neq (1, 1)$.

Known: *One at least of the two statements is true.*

- *e^{π^2} is transcendental.*
- *The two numbers e and π are algebraically independent.*

Other consequence of the sharp five exponentials conjecture: *Transcendence of the number $e^{\lambda^2} = \alpha^{\log \alpha}$ for $\lambda \in \mathcal{L}$, $e^\lambda = \alpha \in \overline{\mathbf{Q}}^\times$.*

Proof. Set $x_1 = y_1 = 1$, $x_2 = y_2 = \lambda$, $\gamma = 1$, $\alpha = 0$, $\beta_{11} = 1$, $\beta_{ij} = 0$ for $(i, j) \neq (1, 1)$.

Other consequence of the sharp five exponentials conjecture: *Transcendence of the number $e^{\lambda^2} = \alpha^{\log \alpha}$ for $\lambda \in \mathcal{L}$, $e^\lambda = \alpha \in \overline{\mathbf{Q}}^\times$.*

Proof. Set $x_1 = y_1 = 1$, $x_2 = y_2 = \lambda$, $\gamma = 1$, $\alpha = 0$, $\beta_{11} = 1$, $\beta_{ij} = 0$ for $(i, j) \neq (1, 1)$.

Known: *One at least of the two numbers*

$$e^{\lambda^2} = \alpha^{\log \alpha}, \quad e^{\lambda^3} = \alpha^{(\log \alpha)^2}$$

is transcendental.

Also a consequence of the sharp six exponentials Theorem!

Strong Five Exponentials Conjecture. *Let x_1, x_2 be \mathbb{Q} -linearly independent and y_1, y_2 be \mathbb{Q} -linearly independent. Then one at least of the five numbers*

$$x_1y_1, x_1y_2, x_2y_1, x_2y_2, x_1/x_2$$

does not belong to $\tilde{\mathcal{L}}$.

9 statements

Four exponentials

sharp

Five exponentials

strong

Six exponentials

9 statements

Four exponentials

Conjecture

sharp

Five exponentials

Theorem

strong

Six exponentials

9 statements

Four exponentials

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Theorem

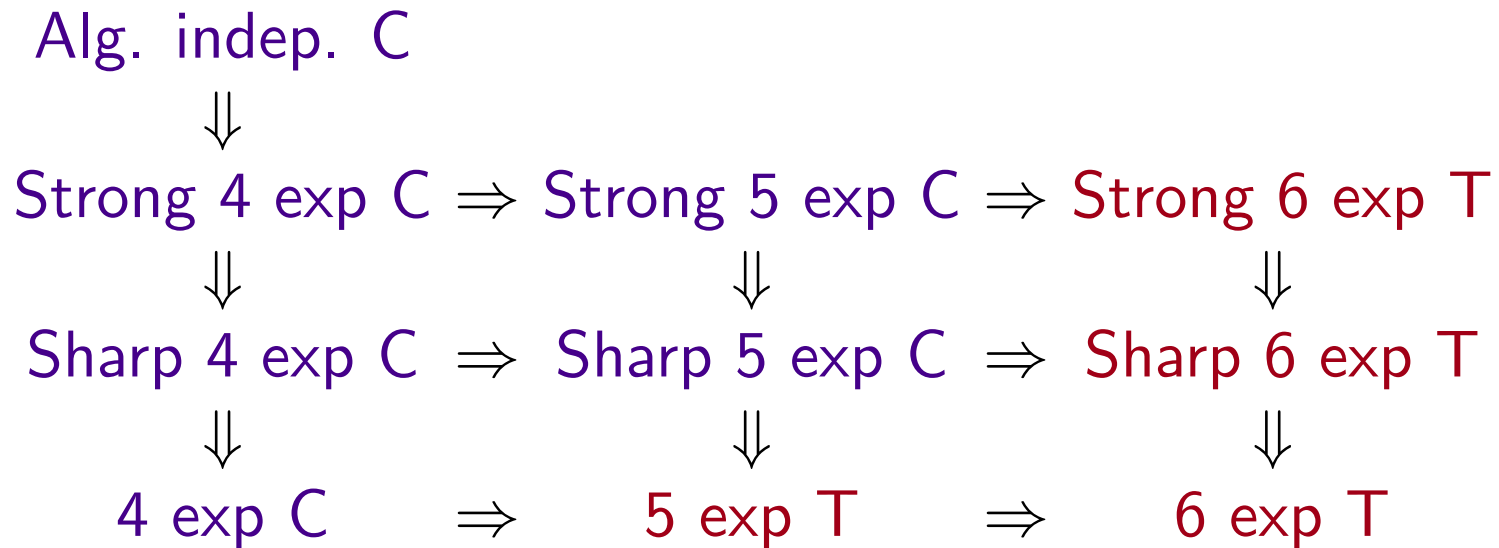
strong

Six exponentials

Four exponentials: three conjectures

Six exponentials: three theorems

Five exponentials: two conjectures (for sharp and strong)
one theorem



Remark. The sharp 6 exponentials Theorem implies the 5 exponentials Theorem.

Consequence of the sharp 4 exponentials Conjecture

Let λ_{ij} ($i = 1, 2, j = 1, 2$) be four non zero logarithms of algebraic numbers.

Assume

$$\lambda_{11} - \frac{\lambda_{12}\lambda_{21}}{\lambda_{22}} \in \overline{\mathbf{Q}}.$$

Then

$$\lambda_{11}\lambda_{22} = \lambda_{12}\lambda_{21}.$$

Proof. Assume

$$\lambda_{11} - \frac{\lambda_{12}\lambda_{21}}{\lambda_{22}} = \beta \in \overline{\mathbf{Q}}.$$

Use the sharp four exponentials conjecture with

$$(\lambda_{11} - \beta)\lambda_{22} = \lambda_{12}\lambda_{21}.$$

Consequence of the strong 4 exponentials Conjecture

Let λ_{ij} ($i = 1, 2, j = 1, 2$) be four non zero logarithms of algebraic numbers.

Assume

$$\frac{\lambda_{11}\lambda_{22}}{\lambda_{12}\lambda_{21}} \in \overline{\mathbf{Q}}.$$

Then

$$\frac{\lambda_{11}\lambda_{22}}{\lambda_{12}\lambda_{21}} \in \mathbf{Q}.$$

Proof: Assume

$$\frac{\lambda_{11}\lambda_{22}}{\lambda_{12}\lambda_{21}} = \beta \in \overline{\mathbb{Q}}.$$

Use the strong four exponentials conjecture with

$$\lambda_{11}\lambda_{22} = \beta\lambda_{12}\lambda_{21}.$$

Consequence of the strong 4 exponentials Conjecture

Let λ_{ij} ($i = 1, 2, j = 1, 2$) be four non zero logarithms of algebraic numbers.

Assume

$$\frac{\lambda_{11}}{\lambda_{12}} - \frac{\lambda_{21}}{\lambda_{22}} \in \overline{\mathbf{Q}}.$$

Then

- either $\lambda_{11}/\lambda_{12} \in \mathbf{Q}$ and $\lambda_{21}/\lambda_{22} \in \mathbf{Q}$
- or $\lambda_{12}/\lambda_{22} \in \mathbf{Q}$ and

$$\frac{\lambda_{11}}{\lambda_{12}} - \frac{\lambda_{21}}{\lambda_{22}} \in \mathbf{Q}.$$

Remark:

$$\frac{\lambda_{11}}{\lambda_{12}} - \frac{b\lambda_{11} - a\lambda_{12}}{b\lambda_{12}} = \frac{a}{b}.$$

Proof: Assume

$$\frac{\lambda_{11}}{\lambda_{12}} - \frac{\lambda_{21}}{\lambda_{22}} = \beta \in \overline{\mathbf{Q}}.$$

Use the strong four exponentials conjecture with

$$\lambda_{12}(\beta\lambda_{22} + \lambda_{21}) = \lambda_{11}\lambda_{22}.$$

Question: Let λ_{ij} ($i = 1, 2, j = 1, 2$) be four non zero logarithms of algebraic numbers. Assume

$$\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21} \in \overline{\mathbf{Q}}.$$

Deduce

$$\lambda_{11}\lambda_{22} = \lambda_{12}\lambda_{21}.$$

Question: Let λ_{ij} ($i = 1, 2, j = 1, 2$) be four non zero logarithms of algebraic numbers. Assume

$$\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21} \in \overline{\mathbb{Q}}.$$

Deduce

$$\lambda_{11}\lambda_{22} = \lambda_{12}\lambda_{21}.$$

Answer: *This is a consequence of the Conjecture on algebraic independence of logarithms of algebraic numbers.*

Consequences of the strong 6 exponentials Theorem

Let λ_{ij} ($i = 1, 2, j = 1, 2, 3$) be six non zero logarithms of algebraic numbers. Assume

- $\lambda_{11}, \lambda_{21}$ are linearly independent over \mathbb{Q}
- and
- $\lambda_{11}, \lambda_{12}, \lambda_{13}$ are linearly independent over \mathbb{Q} .

- *One at least of the two numbers*

$$\lambda_{12} - \frac{\lambda_{11}\lambda_{22}}{\lambda_{21}}, \quad \lambda_{13} - \frac{\lambda_{11}\lambda_{23}}{\lambda_{21}}$$

is transcendental.

- *The same holds for*

$$\frac{\lambda_{12}\lambda_{21}}{\lambda_{11}\lambda_{22}}, \quad \frac{\lambda_{13}\lambda_{21}}{\lambda_{11}\lambda_{23}}$$

- *and for*

$$\frac{\lambda_{12}}{\lambda_{11}} - \frac{\lambda_{22}}{\lambda_{21}}, \quad \frac{\lambda_{13}}{\lambda_{11}} - \frac{\lambda_{23}}{\lambda_{21}}$$

- *Also one at least of the two numbers*

$$\frac{\lambda_{21}}{\lambda_{11}} - \frac{\lambda_{22}}{\lambda_{12}}, \quad \frac{\lambda_{21}}{\lambda_{11}} - \frac{\lambda_{23}}{\lambda_{13}}$$

is transcendental,

- *similarly for*

$$\lambda_{12} - \lambda_{11}\lambda_{22}, \quad \lambda_{13} - \lambda_{11}\lambda_{23}$$

- *and for*

$$\frac{\lambda_{12}}{\lambda_{11}} - \lambda_{22}, \quad \frac{\lambda_{13}}{\lambda_{11}} - \lambda_{23}.$$

Theorem 1. *Let λ_{ij} ($i = 1, 2, j = 1, 2, 3, 4, 5$) be ten non zero logarithms of algebraic numbers. Assume*

- $\lambda_{11}, \lambda_{21}$ are linearly independent over \mathbb{Q}

and

- $\lambda_{11}, \dots, \lambda_{15}$ are linearly independent over \mathbb{Q} .

Then one at least of the four numbers

$$\lambda_{1j}\lambda_{21} - \lambda_{2j}\lambda_{11}, \quad (j = 2, 3, 4, 5)$$

is transcendental.

Elliptic Analogue

Theorem 2. *Let \wp and \wp^* be two non isogeneous Weierstraß elliptic functions with algebraic invariants g_2, g_3 and g_2^*, g_3^* respectively. For $1 \leq j \leq 9$ let u_j (resp. u_j^*) be a non zero logarithm of an algebraic point of \wp (resp. \wp^*). Assume u_1, \dots, u_9 are \mathbb{Q} -linearly independent. Then one at least of the eight numbers*

$$u_j u_1^* - u_j^* u_1 \quad (j = 2, \dots, 9)$$

is transcendental

Motivation: periods of $K3$ surfaces.

Sketch of Proofs

Theorem 3. *Let $G = \mathbf{G}_a^{d_0} \times \mathbf{G}_m^{d_1} \times G_2$ be a commutative algebraic group over $\overline{\mathbf{Q}}$ of dimension $d = d_0 + d_1 + d_2$, V a hyperplane of the tangent space $T_e(G)$, Y a finitely generated subgroup of V of rank ℓ_1 such that $\exp_G(Y) \subset G(\overline{\mathbf{Q}})$ with*

$$\ell_1 > (d - 1)(d_1 + 2d_2).$$

Then V contains a non zero algebraic Lie subalgebra of $T_e(G)$ defined over $\overline{\mathbf{Q}}$.

Sketch of Proof of Theorem 1

Assume

$$\lambda_{1j}\lambda_{21} - \lambda_{2j}\lambda_{11} = \gamma_j \quad \text{for } 1 \leq j \leq \ell_1.$$

Take $G = \mathbf{G}_a \times \mathbf{G}_m^2$, $d_0 = 1$, $d_1 = 2$, $G_2 = \{0\}$, V is the hyperplane

$$z_0 = \lambda_{21}z_1 - \lambda_{11}z_2$$

and $Y = \mathbf{Z}y_1 + \cdots + \mathbf{Z}y_{\ell_1}$ with

$$y_j = (\gamma_j, \lambda_{1j}, \lambda_{2j}), \quad (1 \leq j \leq \ell_1).$$

Since $(d-1)(d_1 + 2d_2) = 4$, we need $\ell_1 \geq 5$.

Sketch of Proof of Theorem 2

Assume

$$u_j u_1^* - u_j^* u_1 = \gamma_j \quad \text{for } 1 \leq j \leq \ell_1.$$

Take $G = \mathbf{G}_a \times \mathcal{E} \times \mathcal{E}^*$, $d_0 = 1$, $d_1 = 0$, $d_2 = 2$, V is the hyperplane

$$z_0 = u_1^* z_1 - u_1 z_2$$

and $Y = \mathbf{Z}y_1 + \cdots + \mathbf{Z}y_{\ell_1}$ with

$$y_j = (\gamma_j, u_j, u_j^*), \quad (1 \leq j \leq \ell_1).$$

Since $(d-1)(d_1 + 2d_2) = 8$, we need $\ell_1 \geq 9$.

In both cases we need to check that V does not contain a non zero Lie subalgebra of $T_e(G)$. For Theorem 1 this follows from the assumption

$\lambda_{11}, \lambda_{21}$ are \mathbb{Q} -linearly independent,

while for Theorem 2 this follows from the assumption

$\mathcal{E}, \mathcal{E}^*$ are not isogeneous.

WIEN, ESI, October 7, 2003

Erwin Schrödinger Institute – Wolfgang Schmidt Conference

Happy Birthday Wolfgang!

<http://www.math.jussieu.fr/~miw/articles/pdf/Wien-ESI.pdf>