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On the abc Conjecture and some of its consequences

by

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Abstract

According to *Nature News*, 10 September 2012, quoting Dorian Goldfeld, the abc Conjecture is "the most important unsolved problem in Diophantine analysis". It is a kind of grand unified theory of Diophantine curves : "The remarkable thing about the abc Conjecture is that it provides a way of reformulating an infinite number of Diophantine problems," says Goldfeld, "and, if it is true, of solving them." Proposed independently in the mid-80s by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6), the abc Conjecture describes a kind of balance or tension between addition and multiplication, formalizing the observation that when two numbers a and b are divisible by large powers of small primes, $a + b$ tends to be divisible by small powers of large primes. The abc Conjecture implies – in a few lines – the proofs of many difficult theorems and outstanding conjectures in Diophantine equations– including Fermat's Last Theorem.

Abstract (continued)

This talk will be at an elementary level, giving a collection of consequences of the abc Conjecture. It will not include an introduction to the Inter-universal Teichmüller Theory of Shinichi Mochizuki.

http://www.kurims.kyoto-u.ac.jp/~motizuki/top-english.html

As simple as abc

American Broadcasting Company

http://fr.wikipedia.org/wiki/American_Broadcasting_Company

Annapurna Base Camp, October 22, 2014

Mt. Annapurna (8091m) is the 10th highest mountain in the world and the journey to its base camp is one of the most popular treks on earth.
http://www.himalayanglacier.com/trekking-in-nepal/160/annapurna-base-camp-trek.htm

The radical of a positive integer

According to the fundamental theorem of arithmetic, any integer $n \geq 2$ can be written as a product of prime numbers :

$$n = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t}.$$

The *radical* (also called *kernel*) $\mathrm{Rad}(n)$ of n is the product of the distinct primes dividing n :

$$\mathrm{Rad}(n) = p_1 p_2 \cdots p_t.$$

$$\mathrm{Rad}(n) \leq n.$$

Examples :

$$\mathrm{Rad}(60\,500) = \mathrm{Rad}(2^2 \cdot 5^3 \cdot 11^2) = 2 \cdot 5 \cdot 11 = 110,$$

$$\mathrm{Rad}(82\,852\,996\,681\,926) = 2 \cdot 3 \cdot 23 \cdot 109 = 15\,042.$$

abc–triples

An abc–triple is a triple of three positive integers a, b, c which are coprime, $a < b$ and that $a + b = c$.

Examples:

$$1 + 2 = 3, \quad 1 + 8 = 9,$$

$$1 + 80 = 81, \quad 4 + 121 = 125,$$

$$2 + 3^{10} \cdot 109 = 23^5, \qquad 11^2 + 3^2 5^6 7^3 = 2^{21} \cdot 23.$$

abc–hits

Following F. Beukers, an abc–hit is an abc–triple such that $\mathrm{Rad}(abc) < c$.

http://www.staff.science.uu.nl/~beuke106/ABCpresentation.pdf

Example: $(1, 8, 9)$ is an abc–hit since $1 + 8 = 9$, $\gcd(1, 8, 9) = 1$ and

$$\mathrm{Rad}(1 \cdot 8 \cdot 9) = \mathrm{Rad}(2^3 \cdot 3^2) = 2 \cdot 3 = 6 < 9.$$

But

$$(2, 16, 18)$$

is not an abc–hit since these three numbers are not coprime.

Some abc–hits

$(1, 80, 81)$ is an abc–hit since $1 + 80 = 81$, $\gcd(1, 80, 81) = 1$ and

$$\mathrm{Rad}(1 \cdot 80 \cdot 81) = \mathrm{Rad}(2^4 \cdot 5 \cdot 3^4) = 2 \cdot 5 \cdot 3 = 30 < 81.$$

$(4, 121, 125)$ is an abc–hit since $4 + 121 = 125$, $\gcd(4, 121, 125) = 1$ and

$$\mathrm{Rad}(4 \cdot 121 \cdot 125) = \mathrm{Rad}(2^2 \cdot 5^3 \cdot 11^2) = 2 \cdot 5 \cdot 11 = 110 < 125.$$

Further abc–hits

- $$(2, 3^{10} \cdot 109, 23^5) = (2, 6\,436\,341, 6\,436\,343)$$

is an abc–hit since $2 + 3^{10} \cdot 109 = 23^5$ and $\mathrm{Rad}(2 \cdot 3^{10} \cdot 109 \cdot 23^5) = 15\,042 < 23^5 = 6\,436\,343.$

- $$(11^2, 3^2 \cdot 5^6 \cdot 7^3, 2^{21} \cdot 23) = (121, 48\,234\,275, 48\,234\,496)$$

is an abc–hit since $11^2 + 3^2 \cdot 5^6 \cdot 7^3 = 2^{21} \cdot 23$ and $\mathrm{Rad}(2^{21} \cdot 3^2 \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 23) = 53\,130 < 2^{21} \cdot 23 = 48\,234\,496.$

- $$(1, 5 \cdot 127 \cdot (2 \cdot 3 \cdot 7)^3, 19^6) = (1, 47\,045\,880, 47\,045\,881)$$

is an abc–hit since $1 + 5 \cdot 127 \cdot (2 \cdot 3 \cdot 7)^3 = 19^6$ and $\mathrm{Rad}(5 \cdot 127 \cdot (2 \cdot 3 \cdot 7)^3 \cdot 19^6) = 5 \cdot 127 \cdot 2 \cdot 3 \cdot 7 \cdot 19 = 506\,730.$

abc–triples and abc–hits

Among $15 \cdot 10^6$ abc–triples with $c < 10^4$, we have 120 abc–hits.

Among $380 \cdot 10^6$ abc–triples with $c < 5 \cdot 10^4$, we have 276 abc–hits.

More abc–hits

$$(1, 3^{16} - 1, 3^{16}) = (1, 43\,046\,720, 43\,046\,721)$$

is an abc–hit.
Proof.

$$\begin{aligned} 3^{16} - 1 &= (3^8 - 1)(3^8 + 1) \\ &= (3^4 - 1)(3^4 + 1)(3^8 + 1) \\ &= (3^2 - 1)(3^2 + 1)(3^4 + 1)(3^8 + 1) \\ &= (3 - 1)(3 + 1)(3^2 + 1)(3^4 + 1)(3^8 + 1) \end{aligned}$$

is divisible by 2^6.
Hence

$$\mathrm{Rad}((3^{16} - 1) \cdot 3^{16}) \le \frac{3^{16} - 1}{2^6} \cdot 2 \cdot 3 < 3^{16}.$$

Infinitely many abc–hits

Proposition. *There are infinitely many abc–hits.*

Take $k \ge 1$, $a = 1$, $c = 3^{2^k}$, $b = c - 1$.

Lemma. 2^{k+2} *divides* $3^{2^k} - 1$.

Proof : Induction on k using

$$3^{2^k} - 1 = (3^{2^{k-1}} - 1)(3^{2^{k-1}} + 1).$$

Consequence :

$$\mathrm{Rad}((3^{2^k} - 1) \cdot 3^{2^k}) \le \frac{3^{2^k} - 1}{2^{k+1}} \cdot 3 < 3^{2^k}.$$

Hence

$$(1, 3^{2^k} - 1, 3^{2^k})$$

is an abc–hit.

Infinitely many abc–hits

This argument shows that there exist infinitely many abc–triples such that

$$c > \frac{1}{6 \log 3} R \log R$$

with $R = \mathrm{Rad}(abc)$.

Question : *Are there abc–triples for which $c > \mathrm{Rad}(abc)^2$?*

We do not know the answer.

Examples

When a, b and c are three positive relatively prime integers satisfying $a + b = c$, define

$$\lambda(a, b, c) = \frac{\log c}{\log \mathrm{Rad}(abc)}.$$

Here are the two largest known values for $\lambda(abc)$

$a + b$	$=$	c	$\lambda(a, b, c)$	authors
$2 + 3^{10} \cdot 109$	$=$	23^5	$1.629912\ldots$	É. Reyssat
$11^2 + 3^2 5^6 7^3$	$=$	$2^{21} \cdot 23$	$1.625990\ldots$	B.M. de Weger

At the date of September 11, 2008, 217 abc triples with $\lambda(a, b, c) \ge 1.4$ were known. http://www.math.unicaen.fr/~nitaj/tableabc.pdf
Since August 1, 2015, 238 are known.

http://www.math.leidenuniv.nl/~desmit/abc/index.php?sort=1

Eric Reyssat : $2 + 3^{10} \cdot 109 = 23^5$

Example of Reyssat $2 + 3^{10} \cdot 109 = 23^5$

$a + b = c$

$$a = 2, \qquad b = 3^{10} \cdot 109, \qquad c = 23^5 = 6\,436\,343,$$

$$\mathrm{Rad}(abc) = \mathrm{Rad}(2 \cdot 3^{10} \cdot 109 \cdot 23^5) = 2 \cdot 3 \cdot 109 \cdot 23 = 15\,042,$$

$$\lambda(a,b,c) = \frac{\log c}{\log \mathrm{Rad}(abc)} = \frac{5 \log 23}{\log 15\,042} \simeq 1.62991.$$

Continued fraction

$$2 + 109 \cdot 3^{10} = 23^5$$

Continued fraction of $109^{1/5}$: $[2; 1, 1, 4, 77733, \dots]$,
approximation : $[2; 1, 1, 4] = 23/9$

$$109^{1/5} = 2.555\,555\,39 \dots$$
$$\frac{23}{9} = 2.555\,555\,55 \dots$$

N. A. Carella. *Note on the ABC Conjecture*
http://arXiv.org/abs/math/0606221

Benne de Weger : $11^2 + 3^2 \cdot 5^6 \cdot 7^3 = 2^{21} \cdot 23$

$$\mathrm{Rad}(2^{21} \cdot 3^2 \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 23) = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 = 53\,130.$$

$$2^{21} \cdot 23 = 48\,234\,496 = (53\,130)^{1.625990\dots}$$

Explicit abc Conjecture

According to S. Laishram and T. N. Shorey, an explicit version, due to A. Baker, of the abc Conjecture, yields

$$c < \mathrm{Rad}(abc)^{7/4}$$

for any abc–triple (a, b, c).

The abc Conjecture

Recall that for a positive integer n, the *radical* of n is

$$\mathrm{Rad}(n) = \prod_{p|n} p.$$

abc **Conjecture**. *Let $\varepsilon > 0$. Then the set of abc triples for which*

$$c > \mathrm{Rad}(abc)^{1+\varepsilon}$$

is finite.

Equivalent statement : *For each $\varepsilon > 0$ there exists $\kappa(\varepsilon)$ such that, if a, b and c in $\mathbf{Z}_{>0}$ are relatively prime and satisfy $a + b = c$, then*

$$c < \kappa(\varepsilon)\mathrm{Rad}(abc)^{1+\varepsilon}.$$

Lower bound for the radical of abc

The abc Conjecture is a **lower bound** for the radical of the product abc :

abc **Conjecture**. *For any $\varepsilon > 0$, there exist $\kappa(\varepsilon)$ such that, if a, b and c are relatively prime positive integers which satisfy $a + b = c$, then*

$$\mathrm{Rad}(abc) > \kappa(\varepsilon)c^{1-\varepsilon}.$$

The abc Conjecture of Œsterlé and Masser

The abc Conjecture resulted from a discussion between J. Œsterlé and D. W. Masser in the mid 1980's.

C.L. Stewart and Yu Kunrui

Best known non conditional result : C.L. Stewart and Yu Kunrui (1991, 2001) :

$$\log c \leq \kappa R^{1/3} (\log R)^3.$$

with $R = \mathrm{Rad}(abc)$:

$$c \leq e^{\kappa R^{1/3} (\log R)^3}.$$

Lucien Szpiro

J. Œsterlé and A. Nitaj proved that the abc Conjecture implies a previous conjecture by L. Szpiro on the conductor of elliptic curves.

Given any $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ such that, for every elliptic curve with minimal discriminant Δ and conductor N,

$$|\Delta| < C(\varepsilon) N^{6+\varepsilon}.$$

Further examples

When a, b and c are three positive relatively prime integers satisfying $a + b = c$, define

$$\varrho(a,b,c) = \frac{\log abc}{\log \mathrm{Rad}(abc)}.$$

Here are the two largest known values for $\varrho(abc)$, found by A. Nitaj.

$a + b$	$=$	c	$\varrho(a,b,c)$
$13 \cdot 19^6 + 2^{30} \cdot 5$	$=$	$3^{13} \cdot 11^2 \cdot 31$	$4.41901\ldots$
$2^5 \cdot 11^2 \cdot 19^9 + 5^{15} \cdot 37^2 \cdot 47$	$=$	$3^7 \cdot 7^{11} \cdot 743$	$4.26801\ldots$

On March 19, 2003, 47 abc triples were known with $0 < a < b < c$, $a + b = c$ and $\gcd(a,b) = 1$ satisfying $\varrho(a,b,c) > 4$.

http://www.math.unicaen.fr/~nitaj/tableszpiro.pdf

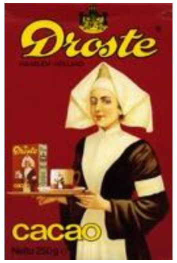
Abderrahmane Nitaj عبدالرحمان نتاج

http://www.math.unicaen.fr/~nitaj/abc.html

Bart de Smit

http://www.math.leidenuniv.nl/~desmit/abc/

Escher and the Droste effect



http://escherdroste.math.leidenuniv.nl/

www.abcathome.com

ABC@home is an educational and non-profit distributed computing project finding abc-triples related to the ABC conjecture.

The ABC conjecture is currently one of the greatest open problems in mathematics. If it is proven to be true, a lot of other open problems can be answered directly from it.

The ABC conjecture is one of the greatest open mathematical questions, one of the holy grails of mathematics. It will teach us something about our very own numbers.

Fermat's Last Theorem $x^n + y^n = z^n$ for $n \geq 6$

Pierre de Fermat
1601 – 1665

Andrew Wiles
1953 –

Solution in 1994

Fermat's last Theorem for $n \geq 6$ as a consequence of the abc Conjecture

Assume $x^n + y^n = z^n$ with $\gcd(x, y, z) = 1$ and $x < y$. Then (x^n, y^n, z^n) is an abc–triple with

$$\mathrm{Rad}(x^n y^n z^n) \leq xyz < z^3.$$

If the explicit abc Conjecture $c < \mathrm{Rad}(abc)^2$ is true, then one deduces

$$z^n < z^6,$$

hence $n \leq 5$ (and therefore $n \leq 2$).

Square, cubes...

- A perfect power is an integer of the form a^b where $a \geq 1$ and $b > 1$ are positive integers.
- Squares :

$$1,\ 4,\ 9,\ 16,\ 25,\ 36,\ 49,\ 64,\ 81,\ 100,\ 121,\ 144,\ 169,\ 196, \ldots$$

- Cubes :

$$1,\ 8,\ 27,\ 64,\ 125,\ 216,\ 343,\ 512,\ 729,\ 1\,000,\ 1\,331, \ldots$$

- Fifth powers :

$$1,\ 32,\ 243,\ 1\,024,\ 3\,125,\ 7\,776,\ 16\,807,\ 32\,768, \ldots$$

Perfect powers

$$1,\ 4,\ 8,\ 9,\ 16,\ 25,\ 27,\ 32,\ 36,\ 49,\ 64,\ 81,\ 100,\ 121,\ 125,\ 128,\ 144,\ 169,\ 196,\ 216,\ 225,\ 243,\ 256,\ 289,\ 324,\ 343,\ 361,\ 400,\ 441,\ 484,\ 512,\ 529,\ 576,\ 625,\ 676,\ 729,\ 784, \ldots$$

Neil J. A. Sloane's encyclopaedia
http://oeis.org/A001597

Nearly equal perfect powers

- Difference 1 : $(8, 9)$
- Difference 2 : $(25, 27), \ldots$
- Difference 3 : $(1, 4)$, $(125, 128), \ldots$
- Difference 4 : $(4, 8)$, $(32, 36)$, $(121, 125), \ldots$
- Difference 5 : $(4, 9)$, $(27, 32)$,...

Two conjectures

Subbayya Sivasankaranarayana Pillai (1901-1950)

Eugène Charles Catalan (1814 – 1894)

- Catalan's Conjecture : In the sequence of perfect powers, $8, 9$ is the only example of consecutive integers.

- Pillai's Conjecture : In the sequence of perfect powers, the difference between two consecutive terms tends to infinity.

Pillai's Conjecture :

- Pillai's Conjecture : In the sequence of perfect powers, the difference between two consecutive terms tends to infinity.

- Alternatively : Let k be a positive integer. The equation

$$x^p - y^q = k,$$

where the unknowns x, y, p and q take integer values, all ≥ 2, has only finitely many solutions (x, y, p, q).

Results

P. Mihăilescu, 2002.

Catalan was right : *the equation* $x^p - y^q = 1$ *where the unknowns* x, y, p *and* q *take integer values, all* ≥ 2, *has only one solution* $(x, y, p, q) = (3, 2, 2, 3)$.

Previous work on Catalan's Conjecture

J.W.S. Cassels, Rob Tijdeman

$$x^p < y^q < \exp\exp\exp\exp(730)$$

Michel Langevin

Previous work on Catalan's Conjecture

Maurice Mignotte Yuri Bilu

Pillai's conjecture and the abc Conjecture

There is no value of $k \geq 2$ for which one knows that Pillai's equation $x^p - y^q = k$ has only finitely many solutions.

Pillai's conjecture as a consequence of the abc Conjecture : *if $x^p \neq y^q$, then*

$$|x^p - y^q| \geq c(\epsilon) \max\{x^p, y^q\}^{\kappa-\epsilon}$$

with

$$\kappa = 1 - \frac{1}{p} - \frac{1}{q}.$$

Not a consequence of the abc Conjecture

$p = 3$, $q = 2$

Hall's Conjecture (1971) :

if $x^3 \neq y^2$, then

$|x^3 - y^2| \geq c \max\{x^3, y^2\}^{1/6}$.

http://en.wikipedia.org/wiki/Marshall_Hall,_Jr

Conjecture of F. Beukers and C.L. Stewart (2010)

Let p, q be coprime integers with $p > q \geq 2$. Then, for any $c > 0$, there exist infinitely many positive integers x, y such that

$$0 < |x^p - y^q| < c \max\{x^p, y^q\}^{\kappa}$$

with $\kappa = 1 - \frac{1}{p} - \frac{1}{q}$.

Generalized Fermat's equation $x^p + y^q = z^r$

Consider the equation $x^p + y^q = z^r$ in positive integers (x, y, z, p, q, r) such that x, y, z relatively prime and p, q, r are ≥ 2.

If
$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \geq 1,$$
then (p, q, r) is a permutation of one of

$$(2, 2, k), \quad (2, 3, 3), \quad (2, 3, 4), \quad (2, 3, 5),$$

$$(2, 4, 4), \quad (2, 3, 6), \quad (3, 3, 3)$$

and in each case the set of solutions (x, y, z) is known (for some of these values there are infinitely many solutions).

Frits Beukers and Don Zagier

For
$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1,$$
10 solutions (x, y, z, p, q, r) (up to obvious symmetries) to the equation
$$x^p + y^q = z^r$$
are known.

On the condition that x, y, z are relatively prime

$$1 + 2^3 = 3^2 \quad \Longrightarrow \quad 3^6 + 18^3 = 3^8$$

.

Starting with $a^p + b^q = c$, multiply by c^{pq} and get
$$(ac^q)^p + (bc^p)^q = c^{pq+1}.$$

`http://mathoverflow.net/`

From Henri Darmon, communicated by Claude Levesque.

Generalized Fermat's equation

For
$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1,$$
the equation
$$x^p + y^q = z^r$$
has the following 10 solutions with x, y, z relatively prime :

$$1 + 2^3 = 3^2, \quad 2^5 + 7^2 = 3^4, \quad 7^3 + 13^2 = 2^9, \quad 2^7 + 17^3 = 71^2,$$

$$3^5 + 11^4 = 122^2, \quad 33^8 + 1\,549\,034^2 = 15\,613^3,$$

$$1\,414^3 + 2\,213\,459^2 = 65^7, \quad 9\,262^3 + 15\,312\,283^2 = 113^7,$$

$$17^7 + 76\,271^3 = 21\,063\,928^2, \quad 43^8 + 96\,222^3 = 30\,042\,907^2.$$

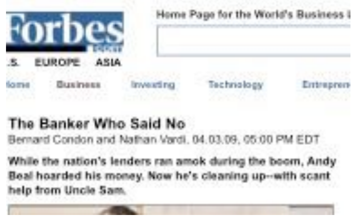
Conjecture of Beal, Granville andTijdeman–Zagier

The equation $x^p + y^q = z^r$ has no solution in positive integers (x, y, z, p, q, r) with each of p, q and r at least 3 and with x, y, z relatively prime.

http://mathoverflow.net/

Andrew Beal

Find a solution with all exponents at least 3, or prove that there is no such solution.



http://www.forbes.com/2009/04/03/banking-andy-beal-business-wall-street-beal.html

Beal's Prize

Mauldin, R. D. – *A generalization of Fermat's last theorem : the Beal Conjecture and prize problem.* Notices Amer. Math. Soc. **44** N°11 (1997), 1436–1437.

The prize. Andrew Beal is very generously offering a prize of \$5,000 for the solution of this problem. The value of the prize will increase by \$5,000 per year up to \$50,000 until it is solved. The prize committee consists of Charles Fefferman, Ron Graham, and R. Daniel Mauldin, who will act as the chair of the committee. All proposed solutions and inquiries about the prize should be sent to Mauldin.

Beal's Prize : $1,000,000$\$ US

An AMS-appointed committee will award this prize for either a proof of, or a counterexample to, the Beal Conjecture published in a refereed and respected mathematics publication. The prize money – currently US\$1,000,000 – is being held in trust by the AMS until it is awarded. Income from the prize fund is used to support the annual Erdős Memorial Lecture and other activities of the Society.

One of Andrew Beal's goals is to inspire young people to think about the equation, think about winning the offered prize, and in the process become more interested in the field of mathematics.

http://www.ams.org/profession/prizes-awards/ams-supported/beal-prize

Henri Darmon, Andrew Granville

"*Fermat-Catalan*" *Conjecture* (H. Darmon and A. Granville), consequence of the abc Conjecture : *the set of solutions* (x, y, z, p, q, r) *to* $x^p + y^q = z^r$ *with* $(1/p) + (1/q) + (1/r) < 1$ *is finite.*

`Hint:` $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ implies $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \le \frac{41}{42}$.

1995 (H. Darmon and A. Granville) : for fixed (p, q, r), only finitely many (x, y, z).

Fermat's Little Theorem

For $a > 1$, any prime p not dividing a divides $a^{p-1} - 1$.

Hence if p is an odd prime, then p divides $2^{p-1} - 1$.

Wieferich primes (1909) : p^2 divides $2^{p-1} - 1$

The only known *Wieferich primes* below $4 \cdot 10^{12}$ are 1093 and 3511.

Infinitely many primes are not Wieferich assuming abc

Joseph H. Silverman

J.H. Silverman : if the abc Conjecture is true, given a positive integer $a > 1$, there exist infinitely many primes p such that p^2 does not divide $a^{p-1} - 1$.

Consecutive integers with the same radical

Notice that

$$75 = 3 \cdot 5^2 \quad \text{and} \quad 1215 = 3^5 \cdot 5$$

hence

$$\mathrm{Rad}(75) = \mathrm{Rad}(1215) = 3 \cdot 5 = 15.$$

But also

$$76 = 2^2 \cdot 19 \quad \text{and} \quad 1216 = 2^6 \cdot 19$$

have the same radical

$$\mathrm{Rad}(76) = \mathrm{Rad}(1216) = 2 \cdot 19 = 38.$$

Consecutive integers with the same radical

For $k \geq 1$, the two numbers

$$x = 2^k - 2 = 2(2^{k-1} - 1)$$

and

$$y = (2^k - 1)^2 - 1 = 2^{k+1}(2^{k-1} - 1)$$

have the same radical, and also

$$x + 1 = 2^k - 1 \quad \text{and} \quad y + 1 = (2^k - 1)^2$$

have the same radical.

Consecutive integers with the same radical

Are there further examples of $x \neq y$ with

$$\mathrm{Rad}(x) = \mathrm{Rad}(y) \quad \text{and} \quad \mathrm{Rad}(x+1) = \mathrm{Rad}(y+1)?$$

Is–it possible to find two distinct integers x, y such that

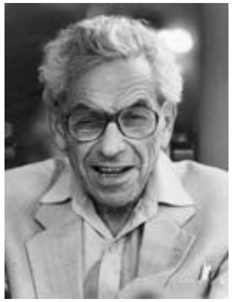
$$\mathrm{Rad}(x) = \mathrm{Rad}(y),$$

$$\mathrm{Rad}(x+1) = \mathrm{Rad}(y+1)$$

and

$$\mathrm{Rad}(x+2) = \mathrm{Rad}(y+2)?$$

Erdős – Woods Conjecture



http://school.maths.uwa.edu.au/~woods/

There exists an absolute constant k such that, if x and y are positive integers satisfying

$$\mathrm{Rad}(x+i) = \mathrm{Rad}(y+i)$$

for $i = 0, 1, \ldots, k-1$, then $x = y$.

Erdős – Woods as a consequence of abc

M. Langevin : The abc Conjecture implies that there exists an absolute constant k such that, if x and y are positive integers satisfying

$$\mathrm{Rad}(x+i) = \mathrm{Rad}(y+i)$$

for $i = 0, 1, \ldots, k-1$, then $x = y$.

Erdős Conjecture on $2^n - 1$

In 1965, P. Erdős conjectured that the greatest prime factor $P(2^n - 1)$ satisfies

$$\frac{P(2^n-1)}{n} \to \infty \quad \text{when} \quad n \to \infty.$$

In 2002, R. Murty and S. Wong proved that this is a consequence of the abc Conjecture.
In 2012, C.L. Stewart proved Erdős Conjecture (in a wider context of Lucas and Lehmer sequences) :

$$P(2^n - 1) > n \exp\bigl(\log n / 104 \log\log n\bigr).$$

Is abc Conjecture optimal ?

Let $\delta > 0$. In 1986, C.L. Stewart and R. Tijdeman proved that there are infinitely many abc–triples for which

$$c > R \exp\left((4-\delta) \frac{(\log R)^{1/2}}{\log\log R} \right).$$

Better than $c > R \log R$.

Conjectures by Machiel van Frankenhuijsen, Olivier Robert, Cam Stewart and Gérald Tenenbaum

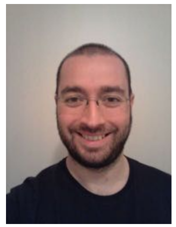
Let $\varepsilon > 0$. There exists $\kappa(\varepsilon) > 0$ such that for any abc triple with $R = \mathrm{Rad}(abc) > 8$,

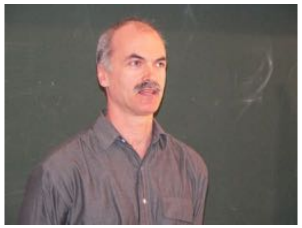
$$c < \kappa(\varepsilon) R \exp\left((4\sqrt{3} + \varepsilon) \left(\frac{\log R}{\log\log R} \right)^{1/2} \right).$$

Further, there exist infinitely many abc–triples for which

$$c > R \exp\left((4\sqrt{3} - \varepsilon) \left(\frac{\log R}{\log\log R} \right)^{1/2} \right).$$

Machiel van Frankenhuijsen, Olivier Robert, Cam Stewart and Gérald Tenenbaum





Heuristic assumption

Whenever a and b are coprime positive integers, $R(a+b)$ is independent of $R(a)$ and $R(b)$.

O. Robert, C.L. Stewart and G. Tenenbaum, *A refinement of the abc conjecture*, Bull. London Math. Soc., Bull. London Math. Soc. (2014) **46** (6) : 1156-1166.
http://blms.oxfordjournals.org/content/46/6/1156.full.pdf

http://iecl.univ-lorraine.fr/~Gerald.Tenenbaum/PUBLIC/Prepublications_et_publications/abc.pdf

Waring's Problem


Edward Waring
(1736 - 1798)

In 1770, a few months before J.L. Lagrange solved a conjecture of Bachet (1621) and Fermat (1640) by proving that every positive integer is the sum of at most four squares of integers, E. Waring wrote :

"Omnis integer numerus vel est cubus, vel e duobus, tribus, 4, 5, 6, 7, 8, vel novem cubis compositus, est etiam quadrato-quadratus vel e duobus, tribus, &, usque ad novemdecim compositus, & sic deinceps"

"Every integer is a cube or the sum of two, three, ...nine cubes ; every integer is also the square of a square, or the sum of up to nineteen such ; and so forth. Similar laws may be affirmed for the correspondingly defined numbers of quantities of any like degree."

Waring's functions $g(k)$ and $G(k)$

• Waring's function g is defined as follows : *For any integer $k \geq 2$, $g(k)$ is the least positive integer s such that any positive integer N can be written $x_1^k + \cdots + x_s^k$.*

• Waring's function G is defined as follows : *For any integer $k \geq 2$, $G(k)$ is the least positive integer s such that any sufficiently large positive integer N can be written $x_1^k + \cdots + x_s^k$.*

$g(k) \geq I(k)$

For each integer $k \geq 2$, define $I(k) = 2^k + \lfloor (3/2)^k \rfloor - 2$. It is easy to show that $g(k) \geq I(k)$ (J. A. Euler, son of Leonhard Euler). Indeed, write

$$3^k = 2^k q + r \quad \text{with} \quad 0 < r < 2^k, \quad q = \lfloor (3/2)^k \rfloor$$

and consider the integer

$$N = 2^k q - 1 = (q-1)2^k + (2^k - 1)1^k.$$

Since $N < 3^k$, writing N as a sum of k-th powers can involve no term 3^k, and since $N < 2^k q$, it involves at most $(q-1)$ terms 2^k, all others being 1^k ; hence it requires a total number of at least $(q-1) + (2^k - 1) = I(k)$ terms.

The ideal Waring's Theorem $g(k) = I(k)$

Conjecture (C.A. Bretschneider, 1853) : $g(k) = I(k)$ for any $k \geq 2$, with

$$I(k) = 2^k + \lfloor (3/2)^k \rfloor - 2.$$

We know that the remainder $r = 3^k - 2^k q$ satisfies $r < 2^k$. A slight improvement of this upper bound would yield the desired result. L.E. Dickson and S.S. Pillai proved independently in 1936 that $g(k) = I(k)$, provided that $r = 3^k - 2^k q$ satisfies

$$r \leq 2^k - q - 2 \quad \text{with} \quad q = \lfloor (3/2)^k \rfloor..$$

The condition $r \leq 2^k - q - 2$ is satisfied for $4 \leq k \leq 471\ 600\ 000$.

$n = x_1^4 + \cdots + x_{19}^4 : g(4) = 19$

Any positive integer is the sum of at most 19 *biquadrates*
R. Balasubramanian, J-M. Deshouillers, F. Dress (1986).

François Dress, R. Balasubramanian, Jean-Marc Deshouillers

Mahler's contribution

- The estimate

$$r \leq 2^k - q - 2$$

is valid for all sufficiently large k.

Kurt Mahler
(1903 - 1988)

Hence the ideal Waring's Theorem

$$g(k) = 2^k + \lfloor (3/2)^k \rfloor - 2$$

holds for all sufficiently large k.

Waring's Problem and the abc Conjecture

S. David : the estimate

$$r \leq 2^k - q - 2$$

for sufficiently large k follows from the abc Conjecture.

S. Laishram : the ideal Waring's Theorem $g(k) = 2^k + \lfloor (3/2)^k \rfloor - 2$ follows from the explicit abc Conjecture.

Conjecture of Alan Baker (1996)

Let (a, b, c) be an abc–triple and let $\epsilon > 0$. Then

$$c \leq \kappa\big(\epsilon^{-\omega} R\big)^{1+\epsilon}$$

where κ is an absolute constant, $R = \mathrm{Rad}(abc)$ and $\omega = \omega(abc)$ is the number of distinct prime factors of abc.

Remark of Andrew Granville : the minimum of the function on the right hand side over $\epsilon > 0$ occurs essentially with $\epsilon = \omega / \log R$. This yields a slightly sharper form of the conjecture :

$$c \leq \kappa R \frac{(\log R)^\omega}{\omega!}.$$

Alan Baker : explicit abc Conjecture (2004)

Let (a, b, c) be an abc–triple. Then

$$c \leq \frac{6}{5} R \frac{(\log R)^\omega}{\omega!}$$

with $R = \mathrm{Rad}(abc)$ the radical of abc and $\omega = \omega(abc)$ the number of distinct prime factors of abc.

Shanta Laishram and Tarlok Shorey

The Nagell–Ljunggren equation is the equation

$$y^q = \frac{x^n - 1}{x - 1}$$

in integers $x > 1$, $y > 1$, $n > 2$, $q > 1$.

This means that in basis x, all the digits of the perfect power y^q are 1.
If the explicit abc–conjecture of Baker is true, then the only solutions are

$$11^2 = \frac{3^5 - 1}{3 - 1}, \quad 20^2 = \frac{7^4 - 1}{7 - 1}, \quad 7^3 = \frac{18^3 - 1}{18 - 1}.$$

The abc conjecture for number fields

Andrea Surroca

Méthodes de transcendance et géométrie diophantienne, A. Surroca, Thèse Université de Paris 6, 2003.

The abc conjecture for number fields (continued)

David Masser

Noam Elkies

http://www.math.harvard.edu/~elkies/

abc Conjecture for number fields (continued)

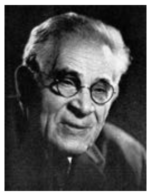
Kálmán Győry
http://www.math.klte.hu/algebra/gyorya.htm

Jerzy Browkin
1934 – 2015

Mordell's Conjecture (Faltings's Theorem)

Using an extension of the abc Conjecture for number fields, N. Elkies deduces Faltings's Theorem on the finiteness of the set of rational points on an algebraic curve of genus ≥ 2.

L.J. Mordell (1922)



G. Faltings (1984)

N. Elkies (1991)

Thue–Siegel–Roth Theorem (Bombieri)

Using the abc Conjecture for number fields, E. Bombieri (1994) deduces a refinement of the Thue–Siegel–Roth Theorem on the rational approximation of algebraic numbers

$$\left|\alpha - \frac{p}{q}\right| > \frac{1}{q^{2+\varepsilon}}$$

where he replaces ε by

$$\kappa(\log q)^{-1/2}(\log\log q)^{-1}$$

where κ depends only on the algebraic number α.

Siegel's zeroes (A. Granville and H.M. Stark)

The uniform *abc* Conjecture for number fields implies a lower bound for the class number of an imaginary quadratic number field, and K. Mahler has shown that this implies that the associated L–function has no Siegel zero.

Further consequences of the *abc* Conjecture

- Erdős's Conjecture on consecutive powerful numbers.
- Dressler's Conjecture : between two positive integers having the same prime factors, there is always a prime.
- Squarefree and powerfree values of polynomials.
- Lang's conjectures : lower bounds for heights, number of integral points on elliptic curves.
- Bounds for the order of the Tate–Shafarevich group.
- Vojta's Conjecture for curves.
- Greenberg's Conjecture on Iwasawa invariants λ and μ in cyclotomic extensions.
- Exponents of class groups of quadratic fields.
- Fundamental units in quadratic and biquadratic fields.

abc and meromorphic function fields

Nevanlinna value distribution theory.

Recent work of Hu, Pei–Chu and Yang, Chung-Chun.

abc and Vojta's height Conjecture

Paul Vojta

Vojta's Conjecture on algebraic points of bounded degree on a smooth complete variety over a global field of characteristic zero implies the *abc* Conjecture.

ABC Theorem for polynomials

Let K be an algebraically closed field. The *radical* of a monic polynomial

$$P(X) = \prod_{i=1}^{n} (X - \alpha_i)^{a_i} \in K[X]$$

with α_i pairwise distinct is defined as

$$\mathrm{Rad}(P)(X) = \prod_{i=1}^{n} (X - \alpha_i) \in K[X].$$

ABC Theorem for polynomials

ABC **Theorem** (A. Hurwitz, W.W. Stothers, R. Mason).
Let A, B, C be three relatively prime polynomials in $K[X]$ with $A + B = C$ and let $R = \mathrm{Rad}(ABC)$. Then

$$\max\{\deg(A), \deg(B), \deg(C)\} < \deg(R).$$

Adolf Hurwitz (1859–1919)

This result can be compared with the abc Conjecture, where the degree replaces the logarithm.

The radical of a polynomial as a gcd

The common zeroes of

$$P(X) = \prod_{i=1}^{n} (X - \alpha_i)^{a_i} \in K[X]$$

and P' are the α_i with $a_i \geq 2$. They are zeroes of P' with multiplicity $a_i - 1$. Hence

$$\mathrm{Rad}(P) = \frac{P}{\gcd(P, P')}.$$

Proof of the ABC Theorem for polynomials

Now suppose $A + B = C$ with A, B, C relatively prime.

Notice that

$$\mathrm{Rad}(ABC) = \mathrm{Rad}(A)\mathrm{Rad}(B)\mathrm{Rad}(C).$$

We may suppose A, B, C to be monic and, say, $\deg(A) \leq \deg(B) \leq \deg(C)$.

Write

$$A + B = C, \qquad A' + B' = C',$$

and

$$AB' - A'B = AC' - A'C.$$

Proof of the ABC Theorem for polynomials

Recall $\gcd(A,B,C)=1$. Since $\gcd(C,C')$ divides $AC'-A'C=AB'-A'B$, it divides also

$$\frac{AB'-A'B}{\gcd(A,A')\gcd(B'B')}$$

which is a polynomial of degree

$$< \deg\big(\mathrm{Rad}(A)\big)+\deg\big(\mathrm{Rad}(B)\big)=\deg\big(\mathrm{Rad}(AB)\big).$$

Hence

$$\deg\big(\gcd(C,C')\big)<\deg\big(\mathrm{Rad}(AB)\big)$$

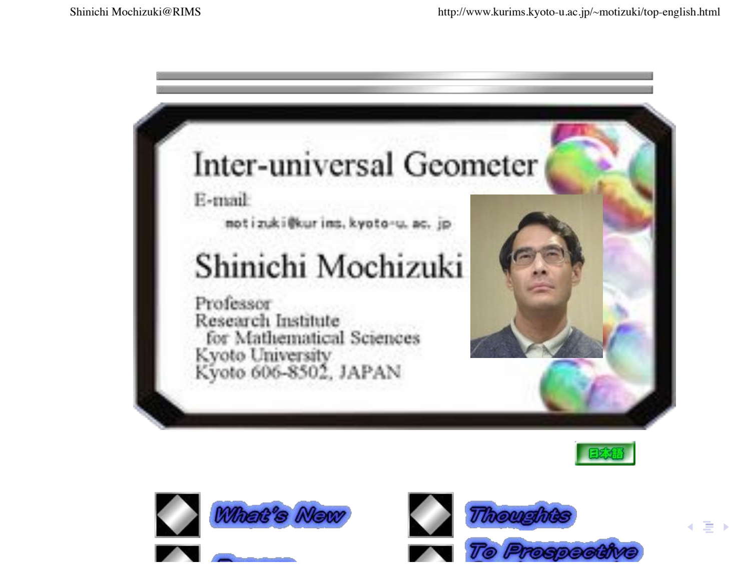
and

$$\deg(C)<\deg\big(\mathrm{Rad}(C)\big)+\deg\big(\mathrm{Rad}(AB)\big)=\deg\big(\mathrm{Rad}(ABC)\big).$$

Shinichi Mochizuki

INTER-UNIVERSAL TEICHMÜLLER THEORY IV : LOG-VOLUME COMPUTATIONS AND SET-THEORETIC FOUNDATIONS by Shinichi Mochizuki

http://www.kurims.kyoto-u.ac.jp/~motizuki/top-english.html



Papers of Shinichi Mochizuki

- General Arithmetic Geometry
- Intrinsic Hodge Theory
- p–adic Teichmüller Theory
- Anabelian Geometry, the Geometry of Categories
- The Hodge-Arakelov Theory of Elliptic Curves
- Inter-universal Teichmüller Theory

Shinichi Mochizuki

[1] Inter-universal Teichmüller Theory I : Construction of Hodge Theaters. PDF

[2] Inter-universal Teichmüller Theory II : Hodge-Arakelov-theoretic Evaluation. PDF

[3] Inter-universal Teichmüller Theory III : Canonical Splittings of the Log-theta-lattice. PDF

[4] Inter-universal Teichmüller Theory IV : Log-volume Computations and Set-theoretic Foundations. PDF

https://en.wikipedia.org/wiki/Abc_conjecture

In August 2012, Shinichi Mochizuki released a series of four preprints containing a serious claim to a proof of the *abc* Conjecture.

When an error in one of the articles was pointed out by Vesselin Dimitrov and Akshay Venkatesh in October 2012, Mochizuki posted a comment on his website acknowledging the mistake, stating that it would not affect the result, and promising a corrected version in the near future. He proceeded to post a series of corrected papers of which the latest dated November 24, 2014.

http://www.kurims.kyoto-u.ac.jp/~motizuki/top-english.html

[1] Inter-universal Teichmuller Theory I: Construction of Hodge Theaters. PDF NEW !! (2013-03-26)

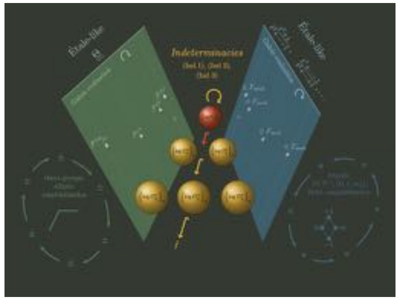
[2] Inter-universal Teichmuller Theory II: Hodge-Arakelov-theoretic Evaluation. PDF NEW !! (2013-02-20)

[3] Inter-universal Teichmuller Theory III: Canonical Splittings of the Log-theta-lattice. PDF NEW !! (2013-04-24

[4] Inter-universal Teichmuller Theory IV: Log-volume Computations and Set-theoretic Foundations. PDF NEW !! (2013-03-26)

[5] A Panoramic Overview of Inter-universal Teichmuller Theory. PDF NEW !! (2013-03-26)

https://www.maths.nottingham.ac.uk/personal/ibf/files/symcor.iut.html



Workshop on IUT Theory of Shinichi Mochizuki, December 7-11 2015

CMI Workshop supported by Clay Math Institute and Symmetries and Correspondences

Organisers : Ivan Fesenko, Minhyong Kim, Kobi Kremnitzer
Finding the speakers and the program of the workshop : Ivan Fesenko

CMI Workshop supported by Clay Math Institute and Symmetries and Correspondences

The work (currently being refereed) of SHINICHI MOCHIZUKI on inter-universal Teichmüller theory (also known as arithmetic deformation theory) and its application to famous conjectures in diophantine geometry became publicly available in August 2012. This theory, developed over 20 years, introduces a vast collection of novel ideas, methods and objects. Aspects of the theory extend arithmetic geometry to a non-scheme-theoretic setting and, more generally, have the potential to open new fundamental areas of mathematics.

The workshop aims to present and analyse key principles, concepts, objects and proofs of the theory of Mochizuki and study its relations with existing theories in different areas, to help to increase the number of experts in the theory of Mochizuki and stimulate its further applications.

Speakers

Shinichi Mochizuki will answer questions during skype sessions of the workshop. He also responds directly to emailed questions.

Invited speakers : Oren Ben-Bassat, Weronika Czerniawska, Yuichiro Hoshi, Ariyan Javanpeykar, Kiran Kedlaya, Robert Kucharczyk, Ulf Kühn, Lars Kuehne, Emmanuel Lepage, Chung Pang Mok, Jakob Stix, Tamás Szamuely, Fucheng Tan, Go Yamashita, Shou-Wu Zhang.

Participants

Participants:
Julio Andrade (Univ. Oxford), Federico Bambozzi (Univ. Regensburg), Alexander Beilinson (Univ. Chicago),
Oren Ben-Bassat (Univ. Haifa), Brian Birch (Univ. Oxford), Francis Brown (Univ. Oxford),
Martin Bridson (Univ. Oxford), Olivia Caramello (Univ. Paris 7), Brian Conrad (Stanford Univ.),
Weronika Czerniawska (Univ. Nottingham), Jahel Dan-Cohen (Univ. Duisburg-Essen),
Jamshid Derakhshan (Univ. Oxford), Taylor Dupuy (Univ. California Los Angeles), Gerd Faltings (MPIM, Bonn),
Ivan Fesenko (Univ. Nottingham), Gerhard Frey (Univ. Duisburg-Essen), Adam Gal (Univ. Oxford),
Lena Gal (Univ. Oxford), Dorian Goldfeld (Columbia Univ.), Nigel Hitchin (Univ. Oxford),
Yuichiro Hoshi (RIMS, Kyoto Univ.), Alexander Ivanov (Techn. Univ. München), Artur Jackson (Purdue Univ.),
Ariyan Javanpeykar (Univ. Mainz), Kiran Kedlaya (Univ. California San Diego), Minhyong Kim (Univ. Oxford),
Kobi Kremnitzer (Univ. Oxford), Robert Kucharczyk (ETH, Zurich), Ulf Kühn (Univ. Hamburg),
Lars Kuehne (MPIM, Bonn), Laurent Lafforgue (IHES, Bures-sur-Yvette), Emmanuel Lepage (Univ. Paris 7),
Junghwan Lim (Univ. Oxford), Angus Macintyre (Univ. Oxford), Nils Matthes (Univ. Hamburg),
Chung Pang Mok (Morningside Center Mathematics Beijing and Purdue Univ.), Alexander Cruz Morales (MPIM, Bonn),
Sergey Oblezin (Univ. Nottingham), Alexander G. Oldenziel (Utrecht Univ.), Thomas Oliver (Univ. Bristol),
Florian Pop (Univ. Pennsylvania at Philadelphia), Damian Rossler (Univ. Oxford),
Thomas Scanlon (Univ. California Berkeley), Francesco Sinkievich (Univ. Oxford), Jakob Stix (Univ. Frankfurt),
Tamás Szamuely (Rényi Inst. Math., Budapest), Fucheng Tan (Shanghai Cent. Math. Sc. & Shanghai Jiao Tong Univ.),
Dinesh Thakur (Rochester Univ.), Ulrike Tillmann (Univ. Oxford), Wester van Urk (Univ. Nottingham),
Felipe Voloch (Univ. Texas Austin), Matthew Walter (Univ. Nottingham), Andrew Wiles (Univ. Oxford),
Bora Yalkinoglu (Univ. Strasbourg), Go Yamashita (RIMS, Kyoto Univ.), Fernando Garcia Yamauti (Univ. Sao Paulo),
Shou-Wu Zhang (Princeton Univ.), Boris Zilber (Univ. Oxford), Lorenzo Lane (Univ. Edinburgh)

Mahidol University International College November 17, 2016

On the abc Conjecture and some of its consequences

by

Michel Waldschmidt

Université P. et M. Curie (Paris VI)

http://www.imj-prg.fr/~michel.waldschmidt/