Algorithmic and algebraic combinatorial aspects of polylogarithms with application on the computation of Drinfel'd associators

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Contents

Polylogarithms and polyzetas occur in

- Control theory (Lille).
- Analysis combinatorics (Flajolet, Labelle, Laforest, Salvy, Vallée, ...).
- Vassiliev knot invariants & Drinfel'd associator (Kontsevich, González-Lorka, Lê, Murakami, Furusho, Racinet, ...).
- Perturbative quantum field theory (Broadhurst, Kreimer, ...).
- Chern classes of a manifold (Hoffman, ...).
- K-theory (Gangl, Wojtkowiak, Zagier, ...).
- Irrationality & transcendence of $\zeta(2k+1)$ (Borwein, Ecalle, Goncharov, Zagier, ...).

Knizhnik–Zamolodchikov equation KZ_3

Drinfel'd constructed the solutions (80s) for the Knizhnik–Zamolodchikov equation KZ_3

$$\frac{dG(z)}{dz} = \frac{1}{2i\pi}\left(\frac{A}{z} + \frac{B}{z-1}\right) G(z), \quad 0 < z < 1,$$

where A, B are noncommuting symbols, and $G(z)$ is a formal power series in A, B with coefficients that are analytic function of z.

$$G_1(z) \sim z^{A/2i\pi} = e^{A/2i\pi \log(z)}, \quad z \to 0,$$
$$G_2(z) \sim (1-z)^{B/2i\pi} = e^{B/2i\pi \log(1-z)}, \quad z \to 1,$$

$$\Phi_{KZ}(A,B) = G_2(z)^{-1} G_1(z).$$
$$\Rightarrow G_1(1-z) \sim z^{B/2i\pi} \Phi_{KZ}(A,B), \quad z \to 0.$$

Question 1 *How to compute the associator* $\Phi_{KZ}(A,B)$ *?*

Drinfel'd associators & $\Phi_{KZ}(A,B)$

Drinfel'd defined associator $\Phi(A,B)$ as a Lie exponential satisfying 3 relations :

1. Duality : $\Phi(B,A) = \Phi^{-1}(A,B)$.
2. Hexagonal : ...
3. Pentagonal : ...

$\Phi_{KZ} \in \mathsf{MZV}\langle\langle A,B\rangle\rangle$ (**Lê & Murakami**), where

$$\mathsf{MZV} = \left\{ \zeta(s_1,\ldots,s_k) = \sum_{n_1>\ldots>n_k>0} \frac{1}{n_1^{s_1}\cdots n_k^{s_k}}. \right\}.$$

Drinfel'd proved also the existence of associators with *rational coefficients*.

Question 2 *How to compute the rational associators ?*

Non-commutative formal power series

X^* : the *free monoide* generated by an alphabet X for the *concatenation* with ϵ (the empty word) as the *neutral* element.

A formal power series S is an infinite sum

$$S = \sum_{w\in X^*} \langle S|w\rangle\, w.$$

A *finite* FPS is called polynomial.

Let $x,y \in X, u,v \in X^*, xu \sqcup\!\sqcup\, yv$ is the polynomial defined recursively as follows

$$xu \sqcup\!\sqcup\, \epsilon = \epsilon \sqcup\!\sqcup\, xu = xu,$$
$$xu \sqcup\!\sqcup\, yv = y[(xu) \sqcup\!\sqcup\, v] + x[u \sqcup\!\sqcup\, (yv)].$$

Example – $x_0x_1 \sqcup\!\sqcup\, x_0x_1 = 4\, x_0^2x_1^2 + 2\, x_0x_1x_0x_1$. □

$\mathbb{C}\langle\langle X\rangle\rangle, \mathbb{C}\langle X\rangle$ denote the sets of FPS and polynomials over X and with coefficients in $\mathbb{C}$.

Operations on formal power series

For $S,T \in \mathbb{C}\langle\langle X\rangle\rangle$, one defines

$$\begin{aligned}
\forall w \in X^*,\ \langle S+T|w\rangle &= \langle S|w\rangle + \langle T|w\rangle,\\
\forall w \in X^*,\quad \langle ST|w\rangle &= \sum_{u,v\in X^*, uv=w} \langle S|v\rangle\langle T|u\rangle,\\
S \sqcup\!\sqcup\, T &= \sum_{u,v\in X^*} \langle S|u\rangle\langle T|v\rangle u \sqcup\!\sqcup\, v.
\end{aligned}$$

$\mathsf{Sh}_{\mathbb{C}}\langle X\rangle$ denotes the polynomial algebra equipped the *shuffle* product $\sqcup\!\sqcup$.

The exponentiel of S is the sum

$$\exp(S) = \sum_{k\geq 0}^{\infty} \frac{S^k}{k!}.$$

The logarithm of $1+S$ is the sum

$$\log(1+S) = \sum_{k\geq 0}^{\infty} (-1)^{k+1}\frac{S^k}{k}.$$

Lyndon words and Standard factorization*

A *Lyndon word* is a non empty word that is less than each of its strict right factors (for the lexicographical ordering).

Example – Let $X = \{x_0,x_1\}, x_0 < x_1$. The Lyndon words of length $\leqslant 5$ are the following (in lexicographically decreasing order) :

$$\begin{aligned}&\{x_0, x_0^4x_1, x_0^3x_1, x_0^3x_1^2, x_0^2x_1, x_0^2x_1x_0x_1, x_0^2x_1^2,\\ &x_0^2x_1^3, x_0x_1, x_0x_1x_0x_1^2, x_0x_1^2, x_0x_1^3, x_0x_1^4, x_1\}\end{aligned}.$$

□

$\mathcal{L}yn(X)$ denotes the set of Lyndon words.

Let $l \in \mathcal{L}yn(X)\backslash X$. A *standard factorization* of l, noted by $\mathsf{st}(l)$, is the *sole* couple (u,v), where u,v are Lyndon words and v is the longest strict right factor of l verifying $u < uv < v$.

Example – $\mathsf{st}(x_0^2x_1x_0x_1) = (x_0^2x_1, x_0x_1)$. □

*M. Lothaire.– *Combinatorics on Words*, Encyclopedia of Mathematics and its Applications, Addison-Wesley, 1983.

Širšov lemma and Radford theorem

Lemma 1 (Širšov) *For any* $w \in X^*$,

$$w = l_1^{i_1} \ldots l_k^{i_k}, \quad l_1 > \ldots > l_k.$$

Example – Let $X = \{x_0, x_1\}, x_0 < x_1$.

$$\begin{aligned} x_1x_0x_1x_1x_0x_1x_1x_0x_0x_1 &= x_1.x_0x_1x_1.x_0x_1x_1.x_0x_0x_1 \\ &= x_1.x_0x_1x_1^2.x_0x_0x_1, \end{aligned}$$

here $x_1 > x_0x_1x_1 > x_0x_0x_1$ are Lyndon words.
□

Theorem 1 (Radford) *The* $\mathbb{C}$*–algebra* $\mathrm{Sh}_{\mathbb{C}}\langle X\rangle$ *is the polynomial algebra generated by* $\mathcal{L}yn(X)$.

Example – Let $X = \{x_0, x_1\}, x_0 < x_1$.

$$\begin{aligned} x_0x_1x_0^2x_1 &= x_0x_1 \sqcup\!\sqcup\, x_0^2x_1 - 3\ x_0^2x_1x_0x_1 - 6\ x_0^3x_1^2, \\ x_0^3x_1x_0^4x_1 &= x_0^3x_1 \sqcup\!\sqcup\, x_0^4x_1 - 5\ x_0^4x_1x_0^3x_1 \\ &\quad -15\ x_0^5x_1x_0^2x_1 - 35\ x_0^6x_1x_0x_1 - 70\ x_0^7x_1^2. \end{aligned}$$

□

Free Lie algebra

The free Lie algebra, noted by $\mathcal{L}ie_{\mathbb{C}}\langle X\rangle$, is the $\mathbb{C}$-algebra of polynomials, over X, equipped the bracket $[.,.]$ defined as follows

$$\forall P, Q \in \mathcal{L}ie_{\mathbb{C}}\langle X\rangle, \quad [P, Q] = PQ - QP$$

and verifying the following properties

$$\begin{aligned} [P, P] &= 0, \\ [P, [Q, R]] + [Q, [R, P]] + [R, [P, Q]] &= 0. \end{aligned}$$

An element of $\mathcal{L}ie_{\mathbb{C}}\langle X\rangle$ is called *Lie polynomial*.

Let $S \in \mathbb{C}\langle\langle X\rangle\rangle$, S is called *Lie series* if it can be written as follows

$$S = \sum_{k \geq 1} P_k,$$

where P_k is a homogenous Lie polynomial of degree k. $\mathcal{L}ie_{\mathbb{C}}\langle\langle X\rangle\rangle$ denotes the set of Lie series over X.

PBW basis and dual basis

The *bracket form* P_l of a Lyndon word l is defined recursively by

$$\begin{cases} P_l = [P_u, P_v] = P_uP_v - P_vP_u & \text{for } uv = \mathrm{st}(l), \\ P_x = x & \text{for } x \in X, \end{cases}$$

The set $\{P_l;\ l \in \mathcal{L}yn(X)\}$ is a basis for the free Lie algebra $\mathcal{L}ie_{\mathbb{C}}\langle X\rangle$.

The PBW basis $\mathcal{B} = \{P_w;\ w \in X^*\}$ is obtained by setting

$$P_v = P_{l_1}^{i_1} P_{l_2}^{i_2} \ldots P_{l_k}^{i_k}, \quad v = l_1^{i_1} \ldots l_k^{i_k}, l_1 > \ldots > l_k,$$

and its dual basis $\mathcal{B}^*$ is obtained by setting

$$\begin{cases} S_l = xS_w, & \forall l \in \mathcal{L}yn(X), l = xw,\ x \in X,\ w \in X^*, \\ S_w = \dfrac{S_{l_1}^{\sqcup\!\sqcup\, i_1} \sqcup\!\sqcup \ldots \sqcup\!\sqcup\, S_{l_k}^{\sqcup\!\sqcup\, i_k}}{i_1! \ldots i_k!}, & w = l_1^{i_1} \ldots l_k^{i_k}, l_1 > \ldots > l_k. \end{cases}$$

Example

l	P_l	$S_l = P_l^* = [l]$
x_0	x_0	x_0
x_1	x_1	x_1
x_0x_1	$[x_0, x_1]$	x_0x_1
$x_0^2x_1$	$[x_0, [x_0, x_1]]$	$x_0^2x_1$
$x_0x_1^2$	$[[x_0, x_1], x_1]$	$x_0x_1^2$
$x_0^3x_1$	$[x_0, [x_0, [x_0, x_1]]]$	$x_0^3x_1$
$x_0^2x_1^2$	$[x_0, [[x_0, x_1], x_1]]$	$x_0^2x_1^2$
$x_0x_1^3$	$[[[x_0, x_1], x_1], x_1]$	$x_0x_1^3$
$x_0^4x_1$	$[x_0, [x_0, [x_0, [x_0, x_1]]]]$	$x_0^4x_1$
$x_0^3x_1^2$	$[x_0, [x_0, [[x_0, x_1], x_1]]]$	$x_0^3x_1^2$
$x_0^2x_1x_0x_1$	$[[x_0, [x_0, x_1]], [x_0, x_1]]$	$2x_0^3x_1^2 + x_0^2x_1x_0x_1$
$x_0^2x_1^3$	$[x_0, [[[x_0, x_1], x_1], x_1]]$	$x_0^2x_1^3$
$x_0x_1x_0x_1^2$	$[[x_0, x_1], [[x_0, x_1], x_1]]$	$3x_0^2x_1^3 + x_0x_1x_0x_1^2$
$x_0x_1^4$	$[[[[x_0, x_1], x_1], x_1], x_1]$	$x_0x_1^4$
$x_0^5x_1$	$[x_0, [x_0, [x_0, [x_0, [x_0, x_1]]]]]$	$x_0^5x_1$
$x_0^4x_1^2$	$[x_0, [x_0, [x_0, [[x_0, x_1], x_1]]]]$	$x_0^4x_1^2$
$x_0^3x_1x_0x_1$	$[x_0, [[x_0, [x_0, x_1]], [x_0, x_1]]]$	$2x_0^4x_1^2 + x_0^3x_1x_0x_1$
$x_0^3x_1^3$	$[x_0, [x_0, [[[x_0, x_1], x_1], x_1]]]$	$x_0^3x_1^3$
$x_0^2x_1x_0x_1^2$	$[x_0, [[x_0, x_1], [[x_0, x_1], x_1]]]$	$3x_0^3x_1^3 + x_0^2x_1x_0x_1^2$
$x_0^2x_1^2x_0x_1$	$[[x_0, [[x_0, x_1], x_1]], [x_0, x_1]]$	$6x_0^3x_1^3 + 3x_0^2x_1x_0x_1^2 + x_0^2x_1^2x_0x_1$
$x_0^2x_1^4$	$[x_0, [[[[x_0, x_1], x_1], x_1], x_1]]$	$x_0^2x_1^4$
$x_0x_1x_0x_1^3$	$[[x_0, x_1], [[[x_0, x_1], x_1], x_1]]$	$4x_0^2x_1^4 + x_0x_1x_0x_1^3$
$x_0x_1^5$	$[[[[[x_0, x_1], x_1], x_1], x_1], x_1]$	$x_0x_1^5$

Factorization Hopf algebra

$\mathbb{C}\langle X\rangle \otimes \mathbb{C}\langle X\rangle$ denotes the tensorial product of $\mathbb{C}\langle X\rangle$ with itself. The co-product Φ of the concatenation is defined as follows

$$\forall u,v \in X^*, \quad \langle \Phi w | u\otimes v\rangle = \langle uv|w\rangle \iff \Phi w = \sum_{u,v\in X^*, uv=w} u\otimes v.$$

Φ is an morphism for the shuffle algebra :

$$\forall u,v\in X^*, \quad \Phi(u \sqcup\!\sqcup v) = \Phi(u) \sqcup\!\sqcup \Phi(v),$$

A co-unity e is defined by :

$$e : \mathbb{C}\langle X\rangle \longrightarrow \mathbb{C}\langle X\rangle, \quad P \longmapsto e(P) = \langle P|\epsilon\rangle.$$

For $S\in\mathbb{C}\langle\langle X\rangle\rangle$, the *antipode* of S is the following FPS ($\tilde{w}$ denotes the *miror* of w)

$$a(S) = \sum_{w\in X^*} (-1)^{|w|}\langle S|w\rangle\, \tilde{w}.$$

$(\mathbb{C}\langle\langle X\rangle\rangle, \sqcup\!\sqcup, 1, \Phi, e, a)$ is the *factorization* Hopf algebra.

Decomposition Hopf algebra

The map Γ_2 is defined as follows

$$\forall u,v,w\in X^*, \quad \langle \Gamma_2 w|u\otimes v\rangle = \langle w|u \sqcup\!\sqcup v\rangle.$$

In particular

$$\forall x\in X, \quad \Gamma_2 x = 1\otimes x + x\otimes 1.$$

It is extended to $\mathbb{C}\langle\langle X\rangle\rangle$ as follows

$$\langle \Gamma_2 S|u\otimes v\rangle = \sum_{w\in X*} \langle S|w\rangle \Gamma_2 w = \langle S|u \sqcup\!\sqcup v\rangle.$$

Γ_2 is an morphism for the associative algebra :

$$\forall u,v\in X^*, \quad \Gamma_2(uv) = \Gamma_2(u)\Gamma_2(v).$$

And it is a co-associative coproduct.

$(\mathbb{C}\langle\langle X\rangle\rangle, ., 1, \Gamma_2, e, a)$ is the *decomposition* Hopf algebra.

Primitive and group-like

Let $S\in\mathbb{C}\langle\langle X\rangle\rangle$, S is called *primitive* if

$$\Gamma_2 S = 1\otimes S + S\otimes 1.$$

S is called *group-like* if

$$\Gamma_2 S = S\otimes S.$$

S verifies the *Friedrichs criterion* if

$$\forall u,v\in X^*, \quad \langle S|u \sqcup\!\sqcup v\rangle = \langle S|u\rangle\langle S|v\rangle.$$

Theorem 2 (Ree)

$S \in \mathcal{L}ie_{\mathbb{C}}\langle\langle X\rangle\rangle$
$\iff$ *S is primitive*
$\iff$ *e^S is group-like*
$\iff$ *e^S verifies the Friedrichs criterion.*

Diagonal series & Schützenberger factorization*

Let us consider, in the completed tensorial product $\mathbb{C}\langle X\rangle\hat{\otimes}\mathbb{C}\langle X\rangle$, the following operation : the shuffle product for the left factor, the concatenation for right factor (for $u_1,v_1,u_2,v_2\in X^*$) :

$$(u_1\otimes v_1)(u_2\otimes v_2) = (u_1 \sqcup\!\sqcup u_2)\otimes(v_1v_2).$$

By a *Schützenberger factorization*, the following diagonal series in $\mathbb{C}\langle X\rangle\hat{\otimes}\mathbb{C}\langle X\rangle$

$$\mathcal{D} = \sum_{w\in X^*} w\otimes w$$

can be factorized in an infinite product, indexed by the Lyndon words :

$$\mathcal{D} = e^{x_1\otimes x_1}\left[\prod_{l\in\mathcal{L}yn(X)\setminus X}^{\searrow} e^{P_l^*\otimes P_l}\right] e^{x_0\otimes x_0}$$

$$\iff \prod_{l\in\mathcal{L}yn(X)\setminus X}^{\searrow} e^{P_l^*\otimes P_l} = e^{-x_1\otimes x_1}\,\mathcal{D}\, e^{-x_0\otimes x_0}.$$

*C. Reutenauer.– *Free Lie Algebras*, London Math. Soc. Monog. 7 (new series), Clarendon Press-Oxford Sciences Publications, 1993.

From iterated integrals to words

The iterated integrals over the differential forms $\{\omega_0(z), \ldots, \omega_n(z)\}$ can be encoded by the words $w = x_{i_1} \ldots x_{i_k}$ over $X = \{x_0, \ldots, x_n\}$ (**Fliess**) :

$$\begin{aligned} \alpha_{z_0}^z(w) &= \int_{z_0}^z \omega_{i_1}(t_1) \ldots \omega_{i_k}(t_k) \\ &= \begin{cases} 1 & \text{if} \quad w = \epsilon, \\ \int_{z_0}^z \omega_i(t) \alpha_{z_0}^t(v) & \text{if} \quad w = x_i v. \end{cases} \end{aligned}$$

Remark – In control theory, Fliess takes the differential forms ω_i that are in the form $a_i(t)dt$, where $a_i(t)$ are real piecewise continous. □

α is a $\mathbb{C}$-algebra morphism for "ш" (**Chen**) :

$\alpha : \mathrm{Sh}_{\mathbb{C}}\langle X \rangle \to \{$comb. of iterated integrals$, +, .\}$.

$$\begin{aligned} \forall u, v \in X^* \setminus \{\epsilon\}, \quad \alpha_{z_0}^z(u + v) &= \alpha_{z_0}^z(u) + \alpha_{z_0}^z(v), \\ \forall \lambda \in \mathbb{C}, u \in X^*, \quad \alpha_{z_0}^z(\lambda u) &= \lambda \alpha_{z_0}^z(u), \\ \forall u, v \in X^*, \quad \alpha_{z_0}^z(u ш v) &= \alpha_{z_0}^z(u) \alpha_{z_0}^z(v). \end{aligned}$$

Iterated integral and shuffle algebra

Let us consider the following differential forms

$$\omega_0(z) = \frac{dz}{z} \quad \text{and} \quad \omega_2(z) = dz.$$

Example – Note that

$$x_0 ш x_2 = x_0 x_2 + x_2 x_0.$$

But $\alpha_0^z(x_0 ш x_2) = \alpha_0^z(x_0)\alpha_0^z(x_2)$ and $\alpha_0^z(x_2 x_0)$ diverge while $\alpha_0^z(x_2) = z$. □

Example – For any $n \geq 0$, one has

$$\alpha_0^z(x_0^n x_2) = \alpha_0^z(x_2) = z.$$

□

Theorem 3 (FPSAC98) *For* $\omega_0 = dz/z, \omega_1 = dz/(1-z)$, α *is* **injective** *from* $\mathrm{Sh}_{\mathbb{C}}\langle x_0, x_1 \rangle$ *to the smallest algebra that contains* $\mathbb{C}$ *and that is stable under integration with respect to* ω_0, ω_1.

From word to polylogarithm

Definition 1 *For any word* $v \in X^* x_1$. *Let us define the* polylogarithms *as follows :*

$$\mathrm{Li}_{x_0 v}(z) = \alpha_0^z(x_0 v) = \int_0^z \omega_0(t) \, \mathrm{Li}_v(t),$$
$$\mathrm{Li}_{x_1 v}(z) = \alpha_0^z(x_1 v) = \int_0^z \omega_1(t) \, \mathrm{Li}_v(t).$$

And for any $v \in x_0 X^* x_1$, *the* polyzetas *as*

$$\zeta(v) = \mathrm{Li}_v(1).$$

Fact 1 *For* $v = x_0^{s_1 - 1} x_1 \ldots x_0^{s_k - 1} x_1$, *one has :*

$$\mathrm{Li}_v(z) = \sum_{n_1 > \ldots > n_k > 0} \frac{z^{n_1}}{n_1^{s_1} \ldots n_k^{s_k}}.$$

Fact 2 $v = x_0^{s_1 - 1} x_1 \ldots x_0^{s_k - 1} x_1 \longleftrightarrow (s_1, \ldots, s_k)$.

$\mathrm{Li}_v(z) = \mathrm{Li}_{s_1, \ldots, s_k}(z)$ *and* $\zeta(v) = \zeta(s_1, \ldots, s_k)$.

We extend the definion 1 over X^* by putting

$$\mathrm{Li}_{x_0^n}(z) = \frac{\log^n(z)}{n!}, \; \mathrm{Li}_{x_1 x_0^n}(z) = \int_0^z \omega_1(t) \frac{\log^n(t)}{n!}.$$

Non-commutative g.s. of $\mathrm{Li}_w(z)$

Definition 2 $L(z) = \sum_{w \in X^*} \mathrm{Li}_w(z) \, w$.

Proposition 1 (FPSAC98) $L(z)$ *satisfies the differential equation (*Drinfel'd equation*) :*

$$dL(z) = [x_0 \omega_0(z) + x_1 \omega_1(z)] L(z)$$

with the boundary condition

$$L(\varepsilon) = e^{x_0 \log \varepsilon} + o(\sqrt{\varepsilon}) \quad \textit{for} \quad \varepsilon \to 0^+.$$

Proof – *(sketched)* Observing that

$$L(z) = 1 + \sum_{u \in X^*} \mathrm{Li}_{x_0 u}(z) x_0 u + \sum_{v \in X^*} \mathrm{Li}_{x_1 v}(z) x_1 v.$$

The exponential term $e^{\log \varepsilon \, x_0}$ comes from the definition of $\mathrm{Li}_{x_0^n}, n \geq 1$.

The coefficient of each other word w in $L(\varepsilon)$ is easily seen to be bounded by $o(\varepsilon^n \log^m \varepsilon)$, where n is the number of x_1's in w. □

Solutions of Drinfel'd equation

Proposition 2 *If $G(z)$ and $H(z)$ are solutions of Drinfel'd equation then*

$$d[H(z)^{-1}G(z)] = 0.$$

Proof – Since $H(z)H(z)^{-1} = 1$ then

$$[dH(z)]H(z)^{-1} = -H(z)[dH(z)^{-1}].$$

Therefore if $H(z)$ is solution then

$$\begin{aligned}[dH(z)^{-1}] &= -H(z)^{-1}[dH(z)]H(z)^{-1} \\ &= -H(z)^{-1}[x_0\omega_0(z) + x_1\omega_1(z)], \\ d[H(z)^{-1}G(z)] &= [dH(z)^{-1}]G(z) + H(z)^{-1}[dG(z)] \\ &= -H(z)^{-1}[x_0\omega_0(z) + x_1\omega_1(z)]G(z) \\ &\quad +H(z)^{-1}[x_0\omega_0(z) + x_1\omega_1(z)]G(z).\end{aligned}$$

We get then the expected result. □

Corollary 1 *Let g_* be the substitution morphism defined by $g_*x_0 = -x_1, g_*x_1 = -x_0$. If $H(z)$ is solution of Drinfel'd equation then*

$$d[H(z)^{-1}g_*H(1-z)] = 0.$$

$L(z)$ is groupe like

Theorem 4 (FPSAC98) $\Delta L(z) = L(z)\otimes L(z)$.

Proof – *(sketched)* Intuitively speaking, it follows from the boundary condition and thus the limit at 0 of $L(z)$ is a Lie exponential, and $L(z)$ is a Lie exponential for any z.

We have to prove $T(z) = \Delta L(z) - L(z) \otimes L(z)$ vanishes for all z. We claim that T satisfies

$$\begin{aligned}dT(z) &= (\Delta V(z))\ T(z)dz, \\ \lim_{\varepsilon\to 0^+} T(\varepsilon) &= 0,\end{aligned}$$

where $V(z) = [x_0\omega_0(z) + x_1\omega_1(z)]$. Thus we have a recursive formula for the coefficients of $T(z)$ by means of differential equations with limit conditions in 0. Since these limits all vanish in 0, it follows by induction that the coefficients of T all vanish globally. □

Factorization of the g.s. $L(z)$

Corollary 2

$$\forall u, v \in X^*,\ \mathrm{Li}_{u \sqcup\!\sqcup v}(z) = \mathrm{Li}_u(z)\,\mathrm{Li}_v(z).$$

Proof – Use the Friedrichs criterion. □

Corollary 3

$$L(z) = e^{-\log(1-z)x_1} L_{\mathrm{reg}}(z) e^{\log(z)x_0},$$

where

$$L_{\mathrm{reg}}(z) = \prod_{l\in\mathcal{L}yn(X)\setminus\{x_0,x_1\}}^{\searrow} \exp(\mathrm{Li}_{P_l^*}(z)P_l).$$

Proof – Use the Schützenberger factorization. □

Asymptotic behaviour at $z = 1$

Corollary 4 *The asymptotic expansion of $L(z)$ at $z = 1$ is given by :*

$$L(1-\varepsilon) \sim e^{-x_1 \log\varepsilon} L_{\mathrm{reg}}(1) e^{x_0\varepsilon}, \quad \textit{for} \quad \varepsilon \to 0^+.$$

Example – For $\varepsilon \to 0^+$, we have

$$\mathrm{Li}_{x_0}(1-\varepsilon) \sim \varepsilon \quad \text{and} \quad \mathrm{Li}_{x_1}(1-\varepsilon) \sim -\log\varepsilon.$$

The Radford theorem gives

$$x_1^2x_0 = x_0x_1^2 - x_0x_1 \sqcup\!\sqcup x_1 + 1/2x_0 \sqcup\!\sqcup x_1^{\sqcup\!\sqcup 2}.$$

Therefore

$$\begin{aligned}\mathrm{Li}_{x_1^2x_0}(1-\varepsilon) &\sim \zeta(2,1) + \zeta(2)\log\varepsilon - \frac{1}{2}\varepsilon\log^2\varepsilon + \ldots \\ &\sim \zeta(3) + \zeta(2)\log\varepsilon - \frac{1}{2}\varepsilon\log^2\varepsilon + \ldots\end{aligned}$$

The last expression is obtained by use of the Euler's identity $\zeta(2,1) = \zeta(3)$. □

In the other words, for any $w \in X^*$, for $\varepsilon \to 0^+$,

$$\mathrm{Li}_w(1-\varepsilon) \sim \sum_{i\geq 1} Q_{w,i}(\log\varepsilon)\varepsilon^i.$$

Non-commutative g.s. of polyzetas

Let $\zeta_{\sqcup\!\sqcup} = \zeta \circ \mathrm{reg}_{\sqcup\!\sqcup}$, where

$$\begin{array}{rl} \mathrm{reg}_{\sqcup\!\sqcup} & : \mathbb{C}\langle\!\langle X\rangle\!\rangle \to \mathbb{C}\langle\!\langle X\rangle\!\rangle, \\ \text{such that} & \mathrm{reg}_{\sqcup\!\sqcup}\, x_0 = \mathrm{reg}_{\sqcup\!\sqcup}\, x_1 = 0, \\ \forall w \in x_0X^*x_1, & \mathrm{reg}_{\sqcup\!\sqcup}\, w = w, \\ \forall u,v \in X^*, & \mathrm{reg}_{\sqcup\!\sqcup}\, u \sqcup\!\sqcup v = \mathrm{reg}_{\sqcup\!\sqcup}\, u \sqcup\!\sqcup \mathrm{reg}_{\sqcup\!\sqcup}\, v. \end{array}$$

Definition 3 $Z = \sum_{w \in \{x_0,x_1\}^*} \zeta_{\sqcup\!\sqcup}(w)\, w.$

Theorem 5 (FPSAC98)

$$Z = L_{\mathrm{reg}}(1) = \prod_{l \in \mathcal{L}yn(X)\setminus\{x_0,x_1\}}^{\searrow} \exp[\zeta(P_l^*)P_l].$$

Proof – Z is the image by $\zeta_{\sqcup\!\sqcup} \otimes id$ of $\mathcal{D}$. □

Corollary 5 $\forall u,v \in X^*, \zeta_{\sqcup\!\sqcup}(u \sqcup\!\sqcup v) = \zeta_{\sqcup\!\sqcup}(u)\zeta_{\sqcup\!\sqcup}(v)$. *Therefore, for any convergent words u and v, $\zeta(u \sqcup\!\sqcup v) = \zeta(u)\zeta(v)$.*

Z and $\log Z$ up to order 4 by computer

$$\begin{aligned} Z = &\cdots e^{\frac{2}{5}\zeta(2)^2[x_0,[x_0,[x_0,x_1]]]} \cdots e^{\zeta(3)[[x_0,x_1],x_1]} \cdots \\ &\cdots e^{\zeta(2)[x_0,x_1]} \cdots e^{\frac{1}{10}\zeta(2)^2[x_0,[[x_0,x_1],x_1]]} \cdots \\ &\cdots e^{\zeta(3)[x_0,[x_0,x_1]]} \cdots e^{\frac{2}{5}\zeta(2)^2[x_0,[x_0,[x_0,x_1]]]} \cdots \\ = &\, 1 + \zeta(2)[x_0,x_1] \\ &+\zeta(3)([x_0,[x_0,x_1]] + [[x_0,x_1],x_1]) \\ &+\frac{2}{5}\zeta(2)^2([x_0,[x_0,[x_0,x_1]]] + [[[x_0,x_1],x_1],x_1 \\ &+\frac{5}{8}[x_0,x_1]^2 + \frac{1}{4}[x_0,[[x_0,x_1],x_1]]) + \cdots, \\ \log Z = &\, \zeta(2)[x_0,x_1] \\ &+\zeta(3)([x_0,[x_0,x_1]] + [[x_0,x_1],x_1]) \\ &+\frac{2}{5}\zeta(2)^2([x_0,[x_0,[x_0,x_1]]] + [[[x_0,x_1],x_1],x_1 \\ &+\frac{1}{4}[x_0,[[x_0,x_1],x_1]]) + \cdots \end{aligned}$$

Chen series & analytic continuation of $L(z)$

For a differentiable path $\gamma : [0,1] \to \mathbb{C}\setminus\{0,1\}$ between a and b, let S_γ be the evaluation at b of the solution of the differential equation

$$\begin{cases} dS_\gamma(z) &= [x_0\omega_0(z) + x_1\omega_1(z)]S_\gamma(z), \\ S_\gamma(a) &= 1. \end{cases}$$

$S_\gamma \in \mathbb{C}\langle X\rangle$ is called the *Chen series* along γ. S_γ is a Lie exponentialand it depends only on the homotopy class of γ (**Chen**).

Proposition 3 (FPSAC98) *Let $z_0 \rightsquigarrow z$ be a differentiable path on $\mathbb{C}\setminus\{0,1\}$ s.t. L admits an analytic continuation. Then $L(z) = S_{z_0 \rightsquigarrow z}L(z_0)$.*

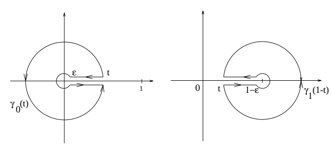
Proof – $L(z)$ and $S_{z_0 \rightsquigarrow z}L(z_0)$ satisfy the Drinfel'd equation taking the same value at z_0. □

Corollary 6

$$S_{\varepsilon \rightsquigarrow 1-\varepsilon} \sim e^{-x_1 \log \varepsilon} Z e^{-x_0 \log \varepsilon} \quad \textit{for} \quad \varepsilon \to 0^+.$$

Proof – $S_{\varepsilon \rightsquigarrow 1-\varepsilon} = L(1-\varepsilon)L(\varepsilon)^{-1}$ and the behaviour of L lead to the expected result. □

Monodromy of the g.s. $L(z)$



Paths of integration

Theorem 6 (FPSAC98) *The monodromy of $L(t)$ for $t \in]0,1[$ around 0 and 1 is given by*

$$\begin{aligned} \mathcal{M}_0 L(t) &= L(t)e^{2i\pi x_0}, \\ \mathcal{M}_1 L(t) &= L(t)Z^{-1}e^{-2i\pi x_1}Z \;=\; L(t)e^{2i\pi \mathfrak{m}_1}, \end{aligned}$$

where $\mathfrak{m}_1$ is a Lie series *given by the formula*

$$\mathfrak{m}_1 = \prod_{l \in \mathcal{L}yn(X)\setminus\{x_0,x_1\}}^{\searrow} e^{-\zeta_{P_l^*}\, \mathrm{ad}\, P_l}(-x_1).$$

Proof of the monodromy theorem

- Monodromy of $L(z)$ around 0

$$\begin{aligned}\mathcal{M}_0 L(t) &= S_{\varepsilon\rightsquigarrow t} S_{\gamma_0(\varepsilon)} S_{t\rightsquigarrow\varepsilon} L(t),\\ &= L(t) L^{-1}(\varepsilon) S_{\gamma_0(\varepsilon)} L(\varepsilon),\\ &= L(t) \lim_{\varepsilon\to 0^+} L^{-1}(\varepsilon) S_{\gamma_0(\varepsilon)} L(\varepsilon),\\ &= L(t) \lim_{\varepsilon\to 0^+} e^{-x_0\log\varepsilon} e^{2i\pi x_0} e^{x_0\log\varepsilon}\\ &= L(t) e^{2i\pi x_0}.\end{aligned}$$

- Monodromy of $L(z)$ around 1

$$\begin{aligned}\mathcal{M}_1 L(t) &= S_{1-\varepsilon\rightsquigarrow t} S_{\gamma_1(\varepsilon)} S_{t\rightsquigarrow 1-\varepsilon} L(t),\\ &= L(t) L^{-1}(1-\varepsilon) S_{\gamma_1(\varepsilon)} L(1-\varepsilon),\\ &= L(t) \lim_{\varepsilon\to 0^+} Z^{-1} e^{x_1\log\varepsilon} e^{-2i\pi x_1} e^{-x_1\log\varepsilon} Z\\ &= L(t) Z^{-1} e^{-2i\pi x_1} Z.\end{aligned}$$

Using the expression of Z and the formula $e^a e^b e^{-a} = e^{e^{\mathrm{ad}_a} b}$, we get finally the expression for $\mathfrak{m}_1$.

The series $\mathfrak{m}_1$ up to order 6 by computer

$$\begin{aligned}\mathfrak{m}_1 \;=\; & -[x_1] + \zeta(x_0x_1)[x_0x_1^2] + \zeta(x_0^2x_1)[x_0^2x_1^2]\\ & +\zeta(x_0x_1^2)[x_0x_1^3] + \zeta(x_0^3x_1)[x_0^3x_1^2]\\ & -\zeta(x_0^3x_1)[x_0^2x_1x_0x_1] + \zeta(x_0^2x_1^2)[x_0^2x_1^3]\\ & +(\zeta(x_0^2x_1^2) - \frac{1}{2}\zeta(x_0x_1)^2)[x_0x_1x_0x_1^2]\\ & +\zeta(x_0x_1^3)[x_0x_1^4] + \zeta(x_0^4x_1)[x_0^4x_1^2]\\ & -2\zeta(x_0^4x_1)[x_0^3x_1x_0x_1] + \zeta(x_0^3x_1^2)[x_0^3x_1^3]\\ & +(3\zeta(x_0^3x_1^2) + \zeta(x_0^2x_1x_0x_1))[x_0^2x_1x_0x_1^2]\\ & +(3\zeta(x_0^3x_1^2) + \zeta(x_0x_1)\zeta(x_0^2x_1)\\ & +2\zeta(x_0^2x_1x_0x_1))[x_0^2x_1^2x_0x_1] + \zeta(x_0^2x_1^3)[x_0^2x_1^4]\\ & +(4\zeta(x_0^2x_1^3) + \zeta(x_0x_1x_0x_1^2))[x_0x_1x_0x_1^3]\\ & +\zeta(x_0x_1^4)[x_0x_1^5]\end{aligned}$$

Monodromy around $z = 1$ (for $p = 2i\pi$)

$$\begin{aligned}\mathcal{M}_1 \operatorname{Li}_{x_0} &= \operatorname{Li}_{x_0}\\ \mathcal{M}_1 \operatorname{Li}_{x_1} &= \operatorname{Li}_{x_1} - p\\ \mathcal{M}_1 \operatorname{Li}_{x_0x_1} &= \operatorname{Li}_{x_0x_1} - p\operatorname{Li}_{x_0}\\ \mathcal{M}_1 \operatorname{Li}_{x_0^2x_1} &= \operatorname{Li}_{x_0^2x_1} - \frac{p}{2}\operatorname{Li}_{x_0}^2\\ \mathcal{M}_1 \operatorname{Li}_{x_0x_1^2} &= \operatorname{Li}_{x_0x_1^2} - p\operatorname{Li}_{x_0x_1} + \frac{p^2}{2}\operatorname{Li}_{x_0} + p\zeta(x_0x_1)\\ \mathcal{M}_1 \operatorname{Li}_{x_0^3x_1} &= \operatorname{Li}_{x_0^3x_1} - \frac{p}{6}\operatorname{Li}_{x_0}^3\\ \mathcal{M}_1 \operatorname{Li}_{x_0^2x_1^2} &= \operatorname{Li}_{x_0^2x_1^2} - p\operatorname{Li}_{x_0^2x_1} + \frac{p^2}{4}\operatorname{Li}_{x_0}^2\\ & \quad + p\zeta(x_0x_1)\operatorname{Li}_{x_0} + p\zeta(x_0^2x_1)\\ \mathcal{M}_1 \operatorname{Li}_{x_0x_1^3} &= \operatorname{Li}_{x_0x_1^3} - p\operatorname{Li}_{x_0x_1^2} + \frac{p^2}{2}\operatorname{Li}_{x_0x_1}\\ & \quad - \frac{p^3}{6}\operatorname{Li}_{x_0} + p\zeta(x_0x_1^2) - \frac{p^2}{2}\zeta(x_0x_1)\\ \mathcal{M}_1 \operatorname{Li}_{x_0^4x_1} &= \operatorname{Li}_{x_0^4x_1} - \frac{p}{24}\operatorname{Li}_{x_0}^4\\ & \vdots\end{aligned}$$

Structure of the monodromy group

Corollary 7 *Monodromy of* Li_w *is given by*

$$\begin{aligned}\forall w \in X^*, \quad \mathcal{M}_0 \operatorname{Li}_{wx_0} &= \operatorname{Li}_{wx_0} + 2i\pi \operatorname{Li}_w + \cdots\\ \mathcal{M}_1 \operatorname{Li}_{wx_1} &= \operatorname{Li}_{wx_1} - 2i\pi \operatorname{Li}_w + \cdots,\end{aligned}$$

The remaining terms are combinations of polylogarithms encoded by words of length $< |w|$.

Proof – The monodromy theorem implies

$M_0 = e^{2i\pi\mathfrak{m}_0} = 1 + 2i\pi x_0 +$ words of length > 1
$M_1 = e^{2i\pi\mathfrak{m}_1} = 1 - 2i\pi x_1 +$ words of length > 1
□

Corollary 8 *The monodromy group of* Li_w *for* $|w| \leqslant n$*is nilpotent at order* $n+1$.

Proof – $M_0 = e^{2i\pi x_0}$ and $M_1 = e^{-2i\pi x_1 + \cdots}$. From $e^A e^B e^{-A} e^{-B} = e^{[A,B]+\cdots}$, it follows that the commutator $M_0 M_1 M_0^{-1} M_1^{-1}$ does not contain any Lie brackets of length 1. Iterating this computation, the brackets of lengths 2, next 3, etc. until n disappear. □

A structure theorem

Theorem 7 (FPSAC98) *The polylogarithms are lenearly indepedant.*

Proof – This is trivial for $n = 0$. Assume that we have proved our assertion for all k, $0 \le k \le n-1$. For $k = n$,

$$\sum_{|w|\leqslant n} \lambda_w \,\mathrm{Li}_w = 0$$
$$\iff \lambda_1 + \sum_{|u|<n} \lambda_{ux_0} \,\mathrm{Li}_{ux_0} + \sum_{|u|<n} \lambda_{ux_1} \,\mathrm{Li}_{ux_1} = 0.$$

(the λ_w are elements of $\mathbb{C}$). Applying $(\mathcal{M}_0 - Id)$ and $(Id - \mathcal{M}_1)$, we have

$$\begin{cases} 2i\pi \sum_{|u|=n-1} \lambda_{ux_0} \,\mathrm{Li}_u + \sum_{|u|<n-1} \mu_u \,\mathrm{Li}_u &= 0, \\ 2i\pi \sum_{|u|=n-1} \lambda_{ux_1} \,\mathrm{Li}_u + \sum_{|u|<n-1} \nu_u \,\mathrm{Li}_u &= 0. \end{cases}$$

By the induction hypothesis, we get the expected result. □

$L(1-t)$

Proposition 4 (FPSAC98) *For any $t \in]0,1[$,*

$$L(1-t) = g_*[L(t)]Z,$$

and g_* is defined by $g_* x_0 = -x_1, g_* x_1 = -x_0$.

Proof – *(sketched)* One has firstly

$$\begin{aligned} L(1-t) &= S_{1-\varepsilon \rightsquigarrow 1-t}\, L(1-\varepsilon) \\ &= g_* S_{\varepsilon \rightsquigarrow t}\, L(1-\varepsilon) \\ &= g_*[L(t) L^{-1}(\varepsilon)]\, L(1-\varepsilon) \\ &= g_* L(t)\; g_* L^{-1}(\varepsilon)\; L(1-\varepsilon). \end{aligned}$$

and secondly

$$L(1-t) \sim g_* L(t)\; g_* e^{-x_0 \log \varepsilon}\, e^{-x_1 \log \varepsilon} Z.$$

If $g(t) = 1-t$ then $g_*\omega_0 = -\omega_1$ and $g_*\omega_1 = -\omega_0$. Therefore $g_* x_0 = -x_1$ and $g_* x_1 = -x_0$. Hence it follows the expected result. □

Corollary 9 *Let g_* be the morphism defined by $g_* x_0 = -x_1$ and $g_* x_1 = -x_0$. Then*

$$L\left(\frac{1}{2}\right) = g_*\left[L\left(\frac{1}{2}\right)\right] Z.$$

$\mathrm{Li}_s(1-t)$ by computer

$$\begin{aligned} \mathrm{Li}_1(1-t) &= -\log(t) \\ \mathrm{Li}_2(1-t) &= -\mathrm{Li}_2(t) + \log(t)\mathrm{Li}_1(t) + \zeta(2) \\ \mathrm{Li}_3(1-t) &= -\mathrm{Li}_{2,1}(t) + \mathrm{Li}_1(t)\mathrm{Li}_2(t) \\ &\quad -\tfrac{1}{2}\log(t)\mathrm{Li}_1(t)^2 \\ &\quad -\zeta(2)\mathrm{Li}_1(t) + \zeta(3) \\ \mathrm{Li}_{2,1}(1-t) &= -\mathrm{Li}_3(t) + \log(t)\mathrm{Li}_2(t) \\ &\quad -\tfrac{1}{2}\log(t)^2\mathrm{Li}_1(t) + \zeta(3) \\ \mathrm{Li}_4(1-t) &= -\mathrm{Li}_{2,1,1}(t) + \mathrm{Li}_1(t)\mathrm{Li}_{2,1}(t) \\ &\quad -\tfrac{1}{2}\mathrm{Li}_1(t)^2\mathrm{Li}_2(t) \\ &\quad +\tfrac{1}{6}\log(t)\mathrm{Li}_1(t)^3 + \tfrac{1}{2}\zeta(2)\mathrm{Li}_1(t)^2 \\ &\quad -\zeta(3)\mathrm{Li}_1(t) + \tfrac{2}{5}\zeta(2)^2 \\ &\;\;\vdots \end{aligned}$$

Duality relation

Proposition 5 *Let τ be the composition of the mirror morphism and of the involutive substitution morphism $x_0 \to x_1$ and $x_1 \to x_0$. Then*

$$Z = \tau(Z).$$

Proof – For $t \in]0,1[$, one has

$$\begin{aligned} S_{t \rightsquigarrow 1-t}(x_0, x_1) &= S_{1-t \rightsquigarrow t}(-x_1, -x_0) \\ &= S^{-1}_{t \rightsquigarrow 1-t}(-x_1, -x_0) \\ &= \tau[S_{t \rightsquigarrow 1-t}(x_0, x_1)]. \end{aligned}$$

By the renormalisation

$$S_{t \rightsquigarrow 1-t} \sim e^{-x_1 \log t} Z e^{-x_0 \log t}, \quad \text{for} \quad t \to 0^+,$$

and then

$$\tau(S_{t \rightsquigarrow 1-t}) \sim e^{-x_1 \log t} \tau(Z) e^{-x_0 \log t}, \quad \text{for} \quad t \to 0^+,$$

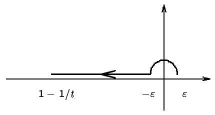
we get the expected result. □

$L(1-1/t)$

Proposition 6 (FPSAC98) *For any* $t \in]0,1[$,

$$L(1-1/t) = g_*[L(t)]g_*(Z^{-1})e^{i\pi x_0},$$

and g_* *is defined by* $g_* x_0 = -x_0 + x_1, g_* x_1 = x_1$.



Proof – $L(1-1/t) = S_{-\varepsilon \rightsquigarrow 1-1/t} S_{\varepsilon \rightsquigarrow -\varepsilon} L(\varepsilon) = S_{-\varepsilon \rightsquigarrow 1-1/t} e^{i\pi x_0} e^{x_0 \log \varepsilon}$. For $g(t) = 1 - 1/t$ then $g_*\omega_0 = -\omega_0 + \omega_1$ and $g_*\omega_1 = -\omega_0$. This leads to $g_* x_0 = -x_0 + x_1$ and $g_* x_1 = -x_0$. Thus

$$\begin{aligned} S_{-\varepsilon \rightsquigarrow 1-1/t} &= g_* S_{1-\varepsilon \rightsquigarrow t} = g_*(L(t)L^{-1}(1-\varepsilon)) \\ &= g_*(L(t)Z^{-1}e^{x_1 \log \varepsilon}). \end{aligned}$$

□

Corollary 10 *For any* $w \in X^*$, *for* $\varepsilon \to 0^+$,

$$\mathrm{Li}_w(-1/\varepsilon) \sim \frac{(-1)^{|w|_{x_0}}}{|w|!} \log^{|w|}(\varepsilon).$$

$\mathrm{Li}_s(1-1/t)$ by computer

$$\begin{aligned} \log(1-1/t) &= (i\pi) - \mathrm{Li}_1(t) - \log(t) \\ \mathrm{Li}_1(1-1/t) &= \log(t) \\ \mathrm{Li}_2(1-1/t) &= \mathrm{Li}_2(t) - \log(t)\mathrm{Li}_1(t) \\ &\quad -\zeta(2) - \tfrac{1}{2}\log(t)^2 \\ \mathrm{Li}_3(1-1/t) &= \mathrm{Li}_{2,1}(t) - \mathrm{Li}_3(t) - \mathrm{Li}_1(t)\mathrm{Li}_2(t) \\ &\quad + \tfrac{1}{2}\log(t)\mathrm{Li}_1(t)^2 \\ &\quad + \left(\zeta(2) + \tfrac{1}{2}\log(t)^2\right)\mathrm{Li}_1(t) \\ &\quad + \log(t)\zeta(2) + \tfrac{1}{6}\log(t)^3 \\ \mathrm{Li}_{2,1}(1-1/t) &= -\mathrm{Li}_3(t) + \log(t)\mathrm{Li}_2(t) \\ &\quad -\tfrac{1}{2}\log(t)^2\mathrm{Li}_1(t) \\ &\quad + \zeta(3) - \tfrac{1}{6}\log(t)^3 \end{aligned}$$

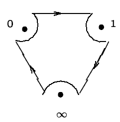
Hexagonal relation

Proposition 7 *Let* ρ *be the substitution morphism* $x_0 \to x_1$ *and* $x_1 \to x_0$. *Then*

$$Ze^{i\pi x_0}\rho(Z)e^{i\pi(-x_0+x_1)}\rho^2(Z)e^{-i\pi x_1} = 1.$$

Proof – Let $g(z) = 1 - 1/z$ permuting the singularities $0, 1$ and ∞. Then $g_*\omega_0 = -\omega_0 + \omega_1$ and $g_*\omega_1 = -\omega_0$. This leads to $g_* x_0 = -x_0 + x_1$ and $g_* x_1 = -x_0$. Thus

$$S_{\varepsilon \rightsquigarrow 1-\varepsilon} e^{i\pi x_0} g_*(S_{\varepsilon \rightsquigarrow 1-\varepsilon} e^{i\pi x_0}) g_*^2(S_{\varepsilon \rightsquigarrow 1-\varepsilon} e^{i\pi x_0}) = 1.$$



By the renormalisation

$$S_{\varepsilon \rightsquigarrow 1-\varepsilon} \sim e^{-x_1 \log \varepsilon} Z e^{-x_0 \log \varepsilon}, \quad \text{for} \quad \varepsilon \to 0^+,$$

we get the expected result. □

By Campbell-Baker-Hausdorf formula, one has

Corollary 11 $\zeta(2) = \pi^2/6$.

Drinfel'd associator $\Phi_{KZ}(A,B)$ and non-commutative g.s. of polyzetas

By changing

$$x_0 := \frac{A}{2i\pi} \quad \text{and} \quad x_1 := -\frac{B}{2i\pi},$$

we have

$$\Phi_{KZ}(A,B) \equiv Z(x_0, x_1)$$

Thus

$$\begin{aligned} \log \Phi_{KZ}(A,B) &= \frac{1}{24}[A,B] \\ &+ \frac{\zeta(3)}{(2i\pi)^3}([[A,B],B] - [A,[A,B]]) \\ &+ \frac{1}{1440}([[[A,B],B],B] - [A,[A,[A,B]]] \\ &\qquad + \frac{1}{4}[A,[[A,B],B]]) + \cdots \end{aligned}$$