Report on some recent advances in Diophantine approximation

Michel Waldschmidt

Université Pierre et Marie Curie–Paris 6, UMR 7586 IMJ Institut de Mathématiques de Jussieu, 4, Place Jussieu, Paris, F–75005 France
miw@math.jussieu.fr
http://www.math.jussieu.fr/~miw/ update: 06/10//2011

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Abstract

A basic question of Diophantine approximation, which is the first issue we discuss, is to investigate the rational approximations to a single real number. Next, we consider the algebraic or polynomial approximations to a single complex number, as well as the simultaneous approximation of powers of a real number by rational numbers with the same denominator. Finally we study generalisations of these questions to higher dimensions. Several recent advances have been made by B. Adamczewski, Y. Bugeaud, S. Fischler, M. Laurent, T. Rivoal, D. Roy and W.M. Schmidt, among others. We review some of these works.

Key words

Diophantine approximation · rational approximation · simultaneous approximation · approximation by algebraic numbers · approximation by linear forms · irrationality measures · transcendence criterion · criteria for algebraic independence · Dirichlet · Hurwitz · Thue–Siegel–Roth–Schmidt · Khintchine · Davenport · Sprindzuck · Laurent · Roy

Introduction

The history of Diophantine approximation is quite old: it includes, for instance, early estimates for π, computations related to astronomical studies, the theory of continued fraction expansion.

There are positive results: *any irrational number has good rational approximations.* One of the simplest tools to see this is Dirichlet's box principle, other methods are continued fraction expansions, Farey series, geometry of numbers (Minkowski's Theorem). There are negative results: *no number has too good (and at the same time too frequent) approximations.* Some results are valid for all (irrational) numbers, others only for restricted classes of numbers, like the class of algebraic numbers. There is a metric theory (§1.2) which deals with almost all numbers in the sense of the Lebesgue measure.

One main goal of the theory of Diophantine approximation is to compare, on the one hand, the distance between a given real number ξ and a rational number p/q, with, on the other hand, the denominator q of the approximant. An approximation is considered as *sharp* if $|\xi - p/q|$ is *small* compared to q. This subject is a classical one, there are a number of surveys, including those by S. Lang [78, 80, 81, 82]. Further general references are [46, 68, 75, 60, 124, 59, 26, 36].

The works by J. Liouville, A. Thue, C.L. Siegel, F.J. Dyson, A.O. Gel'fond, Th. Schneider and K.F. Roth essentially solve the question for the case where ξ is algebraic. In a different direction, a lot of results are known which are valid for almost all numbers, after Khintchine and others.

Several questions arise in this context. One may consider either *asymptotic* or else *uniform* approximation. The former only asks for infinitely many solutions to some inequality, the latter requires that occurrences of such approximations are not too lacunary. As a consequence, one introduces in § 1.1 two exponents for the rational approximation to a single real number ξ, namely $\omega(\xi)$ for the asymptotic approximation and $\widehat{\omega}(\xi)$ for the uniform approximation; a lower bound for such an exponent means that sharp rational approximations exist, an upper bound means that too sharp estimates do no exist. To indicate with a "hat" the exponents of *uniform* Diophantine approximation is a convention which originates in [41].

In this context a new exponent, $\nu(\xi)$, inspired by the pioneer work of R. Apéry in 1976 on $\zeta(3)$, has been introduced recently by T. Rivoal and S. Fischler (§ 1.3).

After rational approximation to a single real number, several other questions naturally arise. One may investigate, for instance, the *algebraic* approximation properties of real or complex numbers, replacing the set of rational numbers by the set of real or complex algebraic numbers. Again, in this context, there are two main points of view: either one considers the distance $|\xi - \alpha|$ between the given real or complex number ξ and algebraic numbers α, or else one investigates the smallness of $|P(\xi)|$ for P a non–zero polynomial with integer coefficients. In both cases there are two parameters, the degree and the height of the algebraic number or of the polynomial, in place of a single one in degree 1, namely q for $\xi - p/q$ or for $P(X) = qX - p$. Algebraic and polynomial approximations are related: on the one hand (Lemma 9), the irreducible polynomial of an algebraic number close to ξ takes a small value at ξ, while on the other hand (Lemma 10), a polynomial taking a small value at ξ is likely to have a root close to ξ. However these connections are not completely understood yet: for instance, while it is easy (by means of Dirichlet's box principle – Lemma 15) to prove the existence of polynomials P having small values $|P(\xi)|$ at the point ξ, it is not so easy to show that sharp algebraic approximations exist (cf. Wirsing's Conjecture 37).

The occurrence of two parameters raises more questions to investigate: often one starts by taking the degree fixed and looking at the behaviour of the approximations as the height tends to infinity; one might do the opposite, fix the height and let the degree tend to infinity: this is is the starting point of a classification of complex numbers by V.G. Sprindžuk (see [36] Chap. 8 p. 166). Another option is to let the sum of the degree and the logarithm of the height tend to infinity: this is the choice of S. Lang who introduced the notion of *size* [79] Chap. V in connection with questions of algebraic independence [78].

The approximation properties of a real or complex number ξ by polynomials of degree at most n (§ 2.3) will give rise to two exponents, $\omega_n(\xi)$ and $\widehat{\omega}_n(\xi)$, which coincide with $\omega(\xi)$ and $\widehat{\omega}(\xi)$ for $n = 1$. Gel'fond's Transcendence Criterion (§ 2.2) is related to an upper bound for the asymptotic exponent of polynomial approximation $\widehat{\omega}_n(\xi)$ valid for all transcendental numbers.

The approximation properties of a real or complex number ξ by algebraic numbers of degree at most n (§ 2.5) will give rise to two further exponents, an asymptotic one $\omega_n^*(\xi)$ and and a uniform one $\widehat{\omega}_n^*(\xi)$, which also coincide with $\omega(\xi)$ and $\widehat{\omega}(\xi)$ for $n = 1$.

For a *real* number ξ, there is a third way of extending the investigation of rational approximation, which is the study of simultaneous approximation by rational numbers of the n-tuple $(\xi, \xi^2, \dots, \xi^n)$. Once more there is an asymptotic exponent $\omega'_n(\xi)$ and a uniform one $\widehat{\omega}'_n(\xi)$, again they coincide with $\omega(\xi)$ and $\widehat{\omega}(\xi)$ for $n = 1$. These two new exponents suffice to describe the approximation properties of a real number by algebraic numbers of degree at most n (the star exponents), thanks to a transference result (Proposition 40) based on the theory of convex bodies of Mahler.

Several relations among these exponents are known, but a number of problems remain open: for instance, for fixed $n \geqslant 1$ the spectrum of the sextuple

$$(\omega_n(\xi),\ \widehat{\omega}_n(\xi),\ \omega'_n(\xi),\ \widehat{\omega}'_n(\xi),\ \omega_n^*(\xi),\ \widehat{\omega}_n^*(\xi)) \in \mathbf{R}^6$$

is far from being completely understood. As we shall see for almost all real numbers ξ and for all algebraic numbers of degree $> n$

$$\omega_n(\xi) = \widehat{\omega}_n(\xi) = \omega_n^*(\xi) = \widehat{\omega}_n^*(\xi) = n \quad \text{and} \quad \omega'_n(\xi) = \widehat{\omega}'_n(\xi) = 1/n.$$

A review of the known properties of these six exponents is given in [41]. We shall repeat some of these facts here (beware that our notation for ω'_n and $\widehat{\omega}'_n$ are the λ_n and $\widehat{\lambda}_n$ from [41], which are the inverse of their w'_n and $\widehat{w}'_n$, also used by Y. Bugeaud in § 3.6 of [36] – here we wish to be compatible with the notation of M. Laurent in [91] for the higher dimensional case).

Among a number of new results in this direction, we shall describe those achieved by D. Roy and others. In particular a number of results for the case $n = 2$ have been recently obtained.

Simultaneous approximations to a tuple of numbers is the next step. The Subspace Theorem of W.M. Schmidt (Theorem 1B in Chapter VI of [124]), which is a powerful generalisation of the Thue-Siegel-Roth Theorem, deals with the approximation of algebraic numbers. It says that tuples of algebraic numbers do behave like almost all tuples. It is a fundamental tool with a number of deep consequences [29]. Another point of view is the metrical one, dealing with almost all numbers. Further questions arise which should concern all tuples and these considerations raise many open problems. We shall report on recent work by M. Laurent who introduces a collection of new exponents for describing the situation.

We discuss these questions mainly in the case of real numbers. Most results (so far as they are not related to the density of $\mathbf{Q}$ into $\mathbf{R}$) are valid also for complex numbers with some modifications (however see [40]), as well and for non–Archimedean valuations, especially p-adic numbers but also (to some extent) for function fields. We make no attempt to be exhaustive, there are a number of related issues which we do not study in detail here – sometimes we just give a selection of recent references. Among them are
• Questions of inhomogeneous approximation.
• Littlewood's Conjecture ([36], Chap. 10).
• Measures of irrationality, transcendence, linear independence, algebraic independence of specific numbers. Effective refinements of Liouville's Theorem are studied in [30] (see also Chap. 2 of [36]).
• Results related to the complexity of the development of irrational algebraic numbers, automata, normality of classical constants (including irrational algebraic numbers) – the Bourbaki lecture by Yu. Bilu [29] on Schmidt's subspace Theorem and its applications describes recent results on this topic and gives further references.
• Connection between Diophantine conditions and dynamical systems.
• Diophantine questions related to Diophantine geometry. Earlier surveys dealing extensively with this issue have been written by S. Lang. A recent reference on this topic is [70].
• In a preliminary version of the present paper, the list of topics which were not covered included also refined results on Hausdorff dimension, Diophantine approximation of dependent quantities and approximation on manifolds, hyperbolic manifolds, also the powerful approach initiated by Dani and Margulis, developed by many specialists. We quote here V.V. Beresnevich, V.I. Bernik, H. Dickinson, M.M. Dodson, D.Y. Kleinbock, È.I. Kovalevskaya, G.A. Margulis, F. Paulin, S.L. Velani. However, thanks to the contribution of Victor Beresnevich and Maurice Dodson who kindly agreed to write sections 2.7 and 3.6 (and also to contribute by adding remarks, especially on § 1.2), these topics are no more excluded.

We discuss only briefly a few questions of algebraic independence; there is much more to say on this matter, especially in connection with Diophantine approximation. Although we quote some recent transcendence criteria as well as criteria for algebraic independence, we do not cover fully the topic (and do not mention criteria for linear independence).

1 Rational approximation to a real number

1.1 Asymptotic and uniform rational approximation: ω and $\widehat{\omega}$

Since $\mathbf{Q}$ is dense in $\mathbf{R}$, *for any* $\xi \in \mathbf{R}$ *and any* $\epsilon > 0$ *there exists* $b/a \in \mathbf{Q}$ *for which*

$$\left|\xi - \frac{b}{a}\right| < \epsilon.$$

Let us write the conclusion

$$|a\xi - b| < \epsilon a.$$

It is easy to improve this estimate: *Let* $a \in \mathbf{Z}_{>0}$ *and let* b *be the nearest integer to* $a\xi$. *Then*

$$|a\xi - b| \leqslant 1/2.$$

A much stronger estimate is due to Dirichlet (1842) and follows from the box or pigeonhole principle – see for instance [78], [60], [124] Chap. I Th. 1A:

Theorem 1 (Uniform Dirichlet's Theorem) *For each real number* $N > 1$, *there exist* q *and* p *in* $\mathbf{Z}$ *with* $1 \leqslant q < N$ *such that*

$$|q\xi - p| < \frac{1}{N}\cdotp$$

As an immediate consequence ([124] Chap. I Cor. 1B):

Corollary 2 (Asymptotic Dirichlet's Theorem) *If* $\xi \in \mathbf{R}$ *is irrational, then there exist infinitely many* $p/q \in \mathbf{Q}$ *for which*

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^2}\cdotp$$

Our first concern is to investigate whether it is possible to improve the uniform estimate of Theorem 1 as well as the asymptotic estimate of Corollary 2.

We start with Corollary 2. Using either the theory of continued fractions or Farey series, one deduces a slightly stronger statement, proved by A. Hurwitz in 1891 (Theorem 2F in Chap. I of [124]):

Theorem 3 (Hurwitz) *For any real irrational number* ξ, *there exist infinitely many* $p/q \in \mathbf{Q}$ *such that*

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}\cdotp$$

For the Golden Ratio $\gamma=(1+\sqrt{5})/2$ and for the numbers related to the Golden Ratio by a homographic transformation $(ax+b)/(cx+d)$ (where a, b, c, d are rational integers satisfying $ad-bc=\pm 1$), this asymptotic result is optimal. For all other irrational real numbers, Hurwitz proved that the constant $\sqrt{5}$ can be replaced by $\sqrt{8}$ ([124] Chap. I Cor. 6C). These are the first elements of the *Lagrange spectrum*: $\sqrt{5}$, $\sqrt{8}$, $\sqrt{221}/5$, $\sqrt{1517}/13$, $\ldots$ (references are given in Chap. I § 6 of [124]; the book [47] is devoted to the Lagrange and Markoff spectra).

Lagrange noticed as early as 1767 (see [59] Chap. 1 Theorem 1.2) that for all irrational quadratic numbers, the exponent 2 in q^2 in the conclusion of Corollary 2 is optimal: more generally, Liouville's inequality (1844) produces, for each algebraic number ξ of degree $d\geqslant 2$, a constant $c(\xi)$ such that, for all rational numbers p/q,

$$\left|\xi-\frac{p}{q}\right|>\frac{c(\xi)}{q^d}\cdot$$

Admissible values for $c(\xi)$ are easy to specify (Th. 1 Chap.1 § 1 of [128], [60] p. 6, Th. 1E of [125], Th. 1.2 of [36]).

A *Liouville number* is a real number ξ for which the opposite estimate holds: *for any $\kappa>0$, there exists a rational number p/q such that*

$$0<\left|\xi-\frac{p}{q}\right|<\frac{1}{q^{\kappa}}\cdot \tag{4}$$

A *very well approximable number* is a real number ξ for which there exists $\kappa>2$ such that the inequality (4) has infinitely many solutions. A nice example of such a number is

$$\xi_\kappa:=2\sum_{n=1}^{\infty}3^{-\lceil\kappa^n\rceil}.$$

for κ a real number >2. This number belongs to the middle third Cantor set $\mathcal{K}$, which is the set of real numbers whose base three expansion are free of the digit 1. In [95], J. Levesley, C. Salp and S.L. Velani show that ξ_κ is an element of $\mathcal{K}$ with irrationality exponent $\mu(\xi_\kappa)=\kappa$ for $\kappa\geqslant(3+\sqrt{5})/2$ and $\geqslant\kappa$ for $2<\kappa\leqslant(3+\sqrt{5})/2$. This example answers a question of K. Mahler on the existence of very well approximable numbers which are not Liouville numbers in $\mathcal{K}$. In [38], Y. Bugeaud shows that $\mu(\xi_\kappa)=\kappa$ for $\kappa\geqslant 2$, and more generally, that for $\kappa\geqslant 2$ and $\lambda>0$, the number

$$\xi_{\lambda,\kappa}:=2\sum_{n=n_0}^{\infty}3^{-\lceil\lambda\kappa^n\rceil}$$

has $\mu(\xi_{\lambda,\kappa})=\kappa$.

Given $\kappa\geqslant 2$, denote by E_κ the set of real numbers ξ satisfying the following property: *the inequality (4) has infinitely many solutions in integers p and q and for any $c<1$ there exists q_0 such that, for $q\geqslant q_0$,*

$$\left|\xi - \frac{p}{q}\right| > \frac{c}{q^\kappa}\cdot \tag{5}$$

Then for any $\kappa \geqslant 2$ this set E_κ is not empty. Explicit examples have been given by Jarník in 1931 (see [2] for a variant). In [20], V.V. Beresnevich, H. Dickinson and S.L. Velani raised the question of the Hausdorff dimension of the set E_κ. The answer is given by Y. Bugeaud in [35]: this dimension is $2/\kappa$.

We now consider the uniform estimate of Theorem 1. Let us show that for any irrational number ξ, Dirichlet's Theorem is essentially optimal: one cannot replace $1/N$ by $1/(2N)$. This was already observed by Khintchine in 1926 [74]:

Lemma 6 *Let ξ be a real number. Assume that there exists a positive integer N_0 such that, for each integer $N \geqslant N_0$, there exist $a \in \mathbf{Z}$ and $b \in \mathbf{Z}$ with $1 \leqslant a < N$ and*

$$|a\xi - b| < \frac{1}{2N}\cdot$$

Then ξ is rational and $a\xi = b$ for each $N \geqslant N_0$.

Proof. By assumption for each integer $N \geqslant N_0$ there exist $a_N \in \mathbf{Z}$ and $b_N \in \mathbf{Z}$ with $1 \leqslant a_N < N$ and

$$|a_N\xi - b_N| < \frac{1}{2N}\cdot$$

Our goal is to check $a_N\xi = b_N$ for each $N \geqslant N_0$.

Let $N \geqslant N_0$. Write (a, b) for (a_N, b_N) and (a', b') for (a_{N+1}, b_{N+1}):

$$|a\xi - b| < \frac{1}{2N}\ (1 \leqslant a \leqslant N-1), \quad |a'\xi - b'| < \frac{1}{2N+2}\ (1 \leqslant a' \leqslant N).$$

Eliminate ξ between $a\xi - b$ and $a'\xi - b'$: the rational integer

$$ab' - a'b = a(b' - a'\xi) + a'(a\xi - b)$$

satisfies $|ab' - a'b| < 1$, hence it vanishes and $ab' = a'b$.

Therefore the rational number $b_N/a_N = b_{N+1}/a_{N+1}$ does not depend on $N \geqslant N_0$. Since

$$\lim_{N\to\infty} b_N/a_N = \xi,$$

it follows that $\xi = b_N/a_N$ for all $N \geqslant N_0$. □

Remark. *As pointed out to me by M. Laurent, an alternative argument is based on continued fraction expansions.*

Coming back to Theorem 1 and Corollary 2, we associate to each real irrational numbers ξ two exponents ω and $\widehat{\omega}$ as follows.

Starting with Corollary 2, we introduce the *asymptotic irrationality exponent of a real number ξ* which is denoted by $\omega(\xi)$:

$$\omega(\xi) = \sup\Big\{w;\ \text{there exist infinitely many } (p,q)\in\mathbf{Z}^2 \\ \text{with } q\geqslant 1 \text{ and } 0<|q\xi-p|\leqslant q^{-w}\Big\}.$$

Some authors prefer to introduce the *irrationality exponent* $\mu(\xi)=\omega(\xi)+1$ *of* ξ which is denoted by $\mu(\xi)$:

$$\mu(\xi) = \sup\Big\{\mu;\ \text{there exist infinitely many } (p,q)\in\mathbf{Z}^2 \\ \text{with } q\geqslant 1 \text{ and } 0<\left|\xi-\frac{p}{q}\right|\leqslant q^{-\mu}\Big\}.$$

An upper bound for $\omega(\xi)$ or $\mu(\xi)$ is an *irrationality measure* for ξ, namely a lower bound for $|\xi-p/q|$ when $p/q\in\mathbf{Q}$.

Liouville numbers are the real numbers ξ with $\omega(\xi)=\mu(\xi)=\infty$.

Since no set E_κ (see property (5)) with $\kappa\geqslant 2$ is empty, the *spectrum* $\big\{\omega(\xi)\ ;\ \xi\in\mathbf{R}\setminus\mathbf{Q}\big\}$ of ω is the whole interval $[1,+\infty]$, while the spectrum $\big\{\mu(\xi)\ ;\ \xi\in\mathbf{R}\setminus\mathbf{Q}\big\}$ of μ is $[2,+\infty]$.

According to the Theorem of Thue–Siegel–Roth [29], *for any real algebraic number* $\xi\in\mathbf{R}\setminus\mathbf{Q}$,

$$\omega(\xi)=1.$$

We shall see (in § 1.2) that the same holds for almost all real numbers.

The other exponent related to Dirichlet's Theorem 1 is the *uniform irrationality exponent of* ξ, denoted by $\widehat{\omega}(\xi)$:

$$\widehat{\omega}(\xi) = \sup\Big\{w;\ \text{for any } N\geqslant 1,\ \text{there exists } (p,q)\in\mathbf{Z}^2 \\ \text{with } 1\leqslant q\leqslant N \text{ and } 0<|q\xi-p|\leqslant N^{-w}\Big\}.$$

In the singular case of a rational number ξ we set $\omega(\xi)=\widehat{\omega}(\xi)=0$. It is plain from the definitions that for any $\xi\in\mathbf{R}\setminus\mathbf{Q}$,

$$\omega(\xi)\geqslant\widehat{\omega}(\xi)\geqslant 1.$$

In fact Lemma 6 implies that *for any* $\xi\in\mathbf{R}\setminus\mathbf{Q}$, $\widehat{\omega}(\xi)=1$. Our motivation to introduce a notation for a number which is always equal to 1 is that it will become non–trivial in more general situations ($\widehat{\omega}_n$ in § 2.3, $\widehat{\omega}'_n$ in § 2.4, $\widehat{\omega}^*_n$ in § 2.5).

1.2 Metric results

The metric theory of Diophantine approximation provides statements which are valid for almost all (real or complex) numbers, that means for all numbers outside a set of Lebesgue measure 0. Among many references on this topic, we quote [129, 130, 26, 69, 36]. See also § 2.7 and § 3.6 below.

One of the early results is due to Capelli: *for almost all* $\xi \in \mathbf{R}$,

$$\omega(\xi) = \widehat{\omega}(\xi) = 1 \quad \textit{and} \quad \mu(\xi) = 2.$$

This is one of many instances where irrational algebraic numbers behave like almost all numbers. However one cannot expect that *all* statements from Diophantine Approximation which are satisfied by all numbers outside a set of measure 0 will be satisfied by all irrational algebraic numbers, just because such an intersection of sets of full measure is empty. As pointed out to me by B. Adamczewski, S. Schanuel (quoted by S. Lang in [78] p. 184 and [83] Chap. II § 2 Th. 6) gave a more precise formulation of such a remark as follows.

Denote by $\mathcal{K}$ (like Khintchine) the set of *non–increasing* functions Ψ from $\mathbf{R}_{\geqslant 1}$ to $\mathbf{R}_{>0}$. Set

$$\mathcal{K}_c = \left\{ \Psi \in \mathcal{K}\,;\, \sum_{n\geqslant 1} \Psi(n) \text{ converges} \right\}, \quad \mathcal{K}_d = \left\{ \Psi \in \mathcal{K}\,;\, \sum_{n\geqslant 1} \Psi(n) \text{ diverges} \right\}$$

Hence $\mathcal{K} = \mathcal{K}_c \cup \mathcal{K}_d$.

A well-known theorem of A. Ya. Khintchine in 1924 (see [73], [75] and Th. 1.10 in [36]) has been refined as follows (an extra condition that the function $x \mapsto x^2\Psi(x)$ is decreasing has been dropped) by Beresnevich, Dickinson and Velani [21]:

Theorem 7 (Khintchine) *Let* $\Psi \in \mathcal{K}$. *Then for almost all real numbers* ξ, *the inequality*

$$|q\xi - p| < \Psi(q) \tag{8}$$

has

- *only finitely many solutions in integers* p *and* q *if* $\Psi \in \mathcal{K}_c$
- *infinitely many solutions in integers* p *and* q *if* $\Psi \in \mathcal{K}_d$.

S. Schanuel proved that the set of real numbers which behave like almost all numbers from the point of view of Khintchine's Theorem in the convergent case has measure 0. More precisely the set of real numbers ξ such that, for any *smooth convex* function $\Psi \in \mathcal{K}_c$, the inequality (8) has only finitely many solutions, is the set of real numbers with bounded partial quotients (*badly approximable numbers* – see [124] Chap. I § 5; other characterisations of this set are given in [83] Chap. II § 2 Th. 6).

Moreover B. Adamczewski and Y. Bugeaud noticed that given any irrational ξ, either there exists a $\Psi \in \mathcal{K}_d$ for which

$$|q\xi - p| < \Psi(q)$$

has no integer solutions or there exists a $\Psi \in \mathcal{K}_c$ for which

$$|q\xi - p| < \Psi(q)$$

has infinitely many integer solutions.

1.3 The exponent ν of S. Fischler and T. Rivoal

Let $\xi \in \mathbf{R}\setminus\mathbf{Q}$. In [63], S. Fischler and T. Rivoal introduce a new exponent $\nu(\xi)$ which they define as follows.

When $\underline{u}=(u_n)_{n\geqslant 1}$ is an increasing sequence of positive integers, define another sequence of integers $\underline{v}=(v_n)_{n\geqslant 1}$ by $|u_n\xi-v_n|<1/2$ (i.e. v_n is the nearest integer to $u_n\xi$) and set

$$\alpha_\xi(\underline{u})=\limsup_{n\to\infty}\frac{|u_{n+1}\xi-v_{n+1}|}{|u_n\xi-v_n|},\quad \beta(\underline{u})=\limsup_{n\to\infty}\frac{u_{n+1}}{u_n}\cdot$$

Then

$$\nu(\xi)=\inf\log\sqrt{\alpha_\xi(\underline{u})\beta(\underline{u})},$$

where $\underline{u}$ ranges over the sequences which satisfy $\alpha_\xi(\underline{u})<1$ and $\beta(\underline{u})<+\infty$. Here we agree that $\inf\emptyset=+\infty$.

They establish a connection with the irrationality exponent by proving:

$$\mu(\xi)\leqslant 1-\frac{\log\beta(\underline{u})}{\log\alpha_\xi(\underline{u})}\cdot$$

As a consequence, *if* $\nu(\xi)<+\infty$*, then* $\mu(\xi)<+\infty$*.*

If ξ is quadratic, Fischler and Rivoal produce a sequence $\underline{u}$ with $\alpha_\xi(\underline{u})\beta(\underline{u})=1$, hence $\nu(\xi)=0$.

This new exponent ν is motivated by Apéry-like proofs of irrationality and measures. Following the works of R. Apéry, A. Baker, F. Beukers, G. Rhin and C. Viola, M. Hata among others, S. Fischler and T. Rivoal deduce

$$\nu(2^{1/3})\leqslant(3/2)\log 2,\quad \nu(\zeta(3))\leqslant 3,\quad \nu(\pi^2)\leqslant 2,\quad \nu(\log 2)\leqslant 1.$$

Also $\nu(\pi)\leqslant 21$.

The spectrum of $\nu(\xi)$ is not yet known. According to [63], *for any* $\xi\in\mathbf{R}\setminus\mathbf{Q}$*, the inequalities* $0\leqslant\nu(\xi)\leqslant+\infty$ *hold.* Further, *for almost all* $\xi\in\mathbf{R}$*,* $\nu(\xi)=0$. Furthermore, S. Fischler and T. Rivoal, completed by B. Adamczewski [1], proved that *any irrational algebraic real number* ξ *has* $\nu(\xi)<+\infty$*.*

There are examples of $\xi\in\mathbf{R}\setminus\mathbf{Q}$ for which $\nu(\xi)=+\infty$, but all known examples with $\nu(\xi)=+\infty$ so far have $\mu(\xi)=+\infty$.

Fischler and Rivoal ask whether *it is true that* $\nu(\xi)<+\infty$ *implies* $\mu(\xi)=2$. Another related question they raise in [63] is whether *there are numbers* ξ *with* $0<\nu(\xi)<+\infty$*.*

2 Polynomial, algebraic and simultaneous approximation to a single number

We define the (usual) height $\mathrm{H}(P)$ of a polynomial

$$P(X) = a_0 + a_1X + \cdots + a_nX^n$$

with complex coefficients as the maximum modulus of its coefficients, while its length $\mathrm{L}(P)$ is the sum of the moduli of these coefficients:

$$\mathrm{H}(P) = \max_{0\leqslant i\leqslant n} |a_i|, \quad \mathrm{L}(P) = \sum_{i=0}^{n} |a_i|.$$

The height $\mathrm{H}(\alpha)$ and length $\mathrm{L}(\alpha)$ of an algebraic number α are the height and length of its minimal polynomial over $\mathbf{Z}$.

2.1 Connections between polynomial approximation and approximation by algebraic numbers

Let ξ be a complex number. To produce a sharp *polynomial approximation* to ξ is to find a non–zero polynomial $P \in \mathbf{Z}[X]$ for which $|P(\xi)|$ is small. An *algebraic approximation* to ξ is an algebraic number α such that the distance $|\xi - \alpha|$ between ξ and α is small. There are close connections between both questions. On the one hand, if $|P(\xi)|$ is small, then ξ is close to a root α of P. On the other hand, if $|\xi - \alpha|$ is small then the minimal polynomial of α assumes a small value at ξ. These connections explain that the classifications of transcendental numbers in S, T and U classes by K. Mahler coincide with the classifications of transcendental numbers in S^*, T^* and U^* classes by J.F. Koksma (see [128] Chap. III and [36] Chap. 3).

The easy part is the next statement (Lemma 15 Chap. III § 3 of [128], § 15.2.4 of [138]), Prop. 3.2 § 3.4 of [36].

Lemma 9 *Let $f \in \mathbf{C}[X]$ be a non–zero polynomial of degree D and length L, let $\alpha \in \mathbf{C}$ be a root of f and let $\xi \in \mathbf{C}$ satisfy $|\xi - \alpha| \leqslant 1$. Then*

$$|f(\xi)| \leqslant |\xi - \alpha| LD(1 + |\xi|)^{D-1}.$$

The other direction requires more work (see Chap. III § 3 of [128], § 3.4 of [36]). The next result is due to G. Diaz and M. Mignotte [52] (cf. Lemma 15.13 of [138]).

Lemma 10 *Let $f \in \mathbf{Z}[X]$ be a non–zero polynomial of degree D. Let ξ be a complex number, α a root of f at minimal distance of ξ and k the multiplicity of α as a root of f. Then*

$$|\xi - \alpha|^k \leqslant D^{3D-2}\mathrm{H}(f)^{2D}|f(\xi)|.$$

Further similar estimates are due to M. Amou and Y. Bugeaud [7].

2.2 Gel'fond's Transcendence Criterion

The so–called *Transcendence Criterion,* proved by A.O. Gel'fond in 1949, is an auxiliary result in the method he introduced in [64, 65, 66] (see also [67] and [68]) for proving algebraic independence results. An example is the algebraic independence of the two numbers $2^{\sqrt[3]{2}}$ and $2^{\sqrt[3]{4}}$. More generally, he proved that *if α is a non–zero algebraic number, $\log\alpha$ a non–zero logarithm of α and β an algebraic number of degree $d \geqslant 3$, then at least 2 among the $d-1$ numbers*

$$\alpha^\beta,\ \alpha^{\beta^2},\ \ldots,\alpha^{\beta^{d-1}}$$

are algebraically independent. Here α^z stands for $\exp(z\log\alpha)$.

While Gel'fond–Schneider transcendence method for solving Hilbert's seventh problem on the transcendence of α^β relies on a *Liouville type* estimate, namely a lower bound for a non–zero value $|P(\xi)|$ of a polynomial P at an algebraic point ξ, Gel'fond's method for algebraic independence requires a more sophisticated result, namely the fact that *there is no non–trivial uniform sequence of polynomials taking small values at a given transcendental number.*

Here is a version of this Transcendence Criterion [131, 93].

Theorem 11 (Gel'fond's Transcendence Criterion) *Let $\xi \in \mathbf{C}$. Assume there is a sequence $(P_N)_{N\geqslant N_0}$ of non–zero polynomials in $\mathbf{Z}[X]$, where P_N has degree $\leqslant N$ and height $\mathrm{H}(P_N) \leqslant e^N$, for which*

$$|P_N(\xi)| \leqslant e^{-6N^2}.$$

Then ξ is algebraic and $P_N(\xi) = 0$ for all $N \geqslant N_0$.

Proof (sketch of). The idea of the proof is basically the same as for Lemma 6 which was dealing with degree 1 polynomials: one eliminates the variable using two consecutive elements of the sequence of polynomials. In degree 1 linear algebra was sufficient. For higher degree the resultant of polynomials is a convenient substitute.

Fix $N \geqslant N_0$. Since $|P_N(\xi)|$ is small, ξ is close to a root α_N of P_N, hence P_N is divisible by a power Q_N of the irreducible polynomial of α_N and $|Q_N(\xi)|$ is small. The resultant of the two polynomials Q_N and Q_{N+1} has absolute value < 1, hence it vanishes and therefore α_N does not depend on N. □

In 1969, H. Davenport and W.M. Schmidt ([49] Theorem 2b) prove the next variant of Gel'fond's Transcendence Criterion, where now the degree is fixed.

Theorem 12 (Davenport and Schmidt) *Let ξ be a real number and $n \geqslant 2$ a positive integer. Assume that for each sufficiently large positive integer N there exists a non–zero polynomial $P_N \in \mathbf{Z}[X]$ of degree $\leqslant n$ and usual height $\leqslant N$ for which*

$$|P_N(\alpha)| \leqslant N^{-2n+1}.$$

Then ξ is algebraic of degree $\leqslant n$.

The next sharp version of Gel'fond's Transcendence Criterion 11, restricted to quadratic polynomials, is due to B. Arbour and D. Roy 2004 [8].

Theorem 13 (Arbour and Roy) *Let ξ be a complex number. Assume that there exists $N_0 > 0$ such that, for any $N \geqslant N_0$, there exists a polynomial $P_N \in \mathbf{Z}[X]$ of degree $\leqslant 2$ and height $\leqslant N$ satisfying*

$$|P_N(\xi)| \leqslant \frac{1}{4} N^{-\gamma-1}.$$

Then ξ is algebraic of degree $\leqslant 2$ and $P_N(\xi) = 0$ for all $N \geqslant N_0$.

Variants of the transcendence criterion have been considered by D. Roy in connection with his new approach towards Schanuel's Conjecture [138] § 15.5.3:

Conjecture 14 (Schanuel) *Let $x_1, \ldots, x_n$ be $\mathbf{Q}$-linearly independent complex numbers. Then n at least of the $2n$ numbers $x_1, \ldots, x_n$, $e^{x_1}, \ldots, e^{x_n}$ are algebraically independent.*

In [111, 112], D. Roy states a Diophantine approximation conjecture, which he shows to be equivalent to Schanuel's Conjecture 14.

Roy's Conjecture involves polynomials for which bounds for the degree, height and absolute values at given points are assumed. The first main difference with Gel'fond's situation is that the smallness of the values is not as strong as what is achieved by Dirichlet's box principle when a single polynomial is constructed. Hence elimination arguments cannot be used without further ideas. On the other hand the assumptions involve not only one polynomial for each n as in Theorem 11, but a collection of polynomials, and they are strongly related. This new situation raises challenging questions on which some advances has already been achieved. In particular in [93] Laurent and Roy obtain variants of Gel'fond's Criterion 11 involving multiplicities. Further progress has been subsequently made by D. Roy in [121, 120].

We shall consider extensions of the Transcendence Criterion to criteria for algebraic independence in § 3.1.

2.3 Polynomial approximation to a single number

A simple application of Dirichlet's box principle (see the proof of Lemma 8.1 in [36]) yields the existence of polynomials with small values at a given real point:

Lemma 15 *Let ξ be a real number and n a positive integer. Set $c=(n+1)\max\{1,|\xi|\}^n$. Then, for each positive integer N, there exists a non–zero polynomial $P\in\mathbf{Z}[X]$, of degree $\leqslant n$ and usual height $\mathrm{H}(P)\leqslant N$, satisfying*

$$|P(\xi)|\leqslant cN^{-n}.$$

Variants of this lemma rely on the geometry of numbers: for instance from Th. B2 in [36] one deduces that in the case $0<|\xi|<1/2$, if $N\geqslant 2$, then the conclusion holds also with $c=1$ (see the proof of Prop. 3.1 in [36]).

Theorem 1 in §1 yields a refined estimate for the special case $n=1$. The statement is plain in the case where ξ is algebraic of degree $\leqslant n$ as soon as N exceeds the height of the irreducible polynomial of ξ – this is why when n is fixed we shall most often assume that ξ is either transcendental or else algebraic of degree $>n$.

The fact that the exponent n in Lemma 15 cannot be replaced by a larger number (even if we ask only such a solution for infinitely many N) was proved by Sprindžuk [129], who showed in 1965 that *for ξ outside a set of Lebesgue measure zero and for each $\epsilon>0$, there are only finitely many non–zero integer polynomials of degree at most n with*

$$|P(\xi)|\leqslant \mathrm{H}(P)^{-n-\epsilon}.$$

We introduce, for each positive integer n and each real number ξ, two exponents $\omega_n(\xi)$ and $\widehat{\omega}_n(\xi)$ as follows.

The number $\omega_n(\xi)$ denotes the supremum of the real numbers w for which there exist infinitely many positive integers N for which the system of inequalities

$$0<|x_0+x_1\xi+\cdots+x_n\xi^n|\leqslant N^{-w},\quad \max_{0\leqslant i\leqslant n}|x_i|\leqslant N \tag{16}$$

has a solution in rational integers $x_0,x_1,\ldots,x_n$. The inequalities (16) can be written

$$0<|P(\xi)|\leqslant \mathrm{H}(P)^{-w}$$

where P denotes a non–zero polynomial with integer coefficients and degree $\leqslant n$. A *transcendence measure* for ξ is a lower bound for $|P(\xi)|$ in terms of the height $\mathrm{H}(P)$ and the degree $\deg P$ of P. Hence one can view an upper bound for $\omega_n(\xi)$ as a transcendence measure for ξ.

These numbers arise in Mahler's classification of complex numbers ([128] Chap. III § 1 and [36] § 3.1).

A uniform version of this exponent is the supremum $\widehat{\omega}_n(\xi)$ of the real numbers w such that, for any sufficiently large integer N, the same system (16) has a solution. An upper bound for $\widehat{\omega}_n(\xi)$ is a *uniform transcendence measure* for ξ.

Clearly, from the definitions, we see that these exponent generalize the ones from §1.1: for $n=1$, $\omega_1(\xi)=\omega(\xi)$ and $\widehat{\omega}_1(\xi)=\widehat{\omega}(\xi)$. From Lemma 15 one deduces, for any $n \geqslant 1$ and any $\xi \in \mathbf{R}$ which is not algebraic of degree $\leqslant n$,

$$n \leqslant \widehat{\omega}_n(\xi) \leqslant \omega_n(\xi). \tag{17}$$

Moreover, $\omega_n \leqslant \omega_{n+1}$ and $\widehat{\omega}_n \leqslant \widehat{\omega}_{n+1}$. As a consequence, Liouville numbers have $\omega_n(\xi) = +\infty$ for all $n \geqslant 1$.

The value of the exponents ω_n and $\widehat{\omega}_n$ for almost all real numbers and for all algebraic numbers of degree $> n$ is n. The following metric result is due to V.G. Sprindžuk [129]:

Theorem 18 (Sprindžuk) *For almost all numbers $\xi \in \mathbf{R}$,*

$$\omega_n(\xi) = \widehat{\omega}_n(\xi) = n \textit{ for all } n \geqslant 1.$$

As a consequence of W.M. Schmidt's subspace Theorem one deduces (see [36] Th. 2.8 and 2.9) the value of $\omega_n(\xi)$ and $\widehat{\omega}_n(\xi)$ for ξ algebraic irrational:

Theorem 19 (Schmidt) *Let $n \geqslant 1$ be an integer and ξ an algebraic number of degree $d > n$. Then*

$$\omega_n(\xi) = \widehat{\omega}_n(\xi) = n.$$

The spectrum of the exponent ω_n is $[n, +\infty]$. For $\widehat{\omega}_n$ it is not completely known. From Theorem 12 one deduces:

Theorem 20 (Davenport and Schmidt) *For any real number ξ which is not algebraic of degree $\leqslant n$,*

$$\widehat{\omega}_n(\xi) \leqslant 2n - 1.$$

For the special case $n = 2$ a sharper estimate holds: from Theorem 13 of B. Arbour and D. Roy one deduces

$$\widehat{\omega}_2(\xi) \leqslant \gamma + 1 \tag{21}$$

(recall that γ denotes the Golden Ratio $(1 + \sqrt{5})/2$).

In [49], Davenport and Schmidt comment:

> *"We have no reason to think that the exponents in these theorems are best possible."*

It was widely believed during a while that $\widehat{\omega}_2(\xi)$ would turn out to be equal to 2 for all $\xi \in \mathbf{R}$ which are not rational nor quadratic irrationals. Indeed, otherwise, for κ in the interval $2 < \kappa < \widehat{\omega}_2(\xi)$, the inequalities

$$0 < |x_0 + x_1\xi + x_2\xi^2| \leqslant cN^{-\kappa}, \quad \max\{|x_0|,\ |x_1|,\ |x_2|\} \leqslant N \tag{22}$$

would have, for a suitable constant $c > 0$ and for all sufficiently large N, a non–trivial solution in integers $(x_0, x_1, x_2) \in \mathbf{Z}^3$. However these inequalities define a convex body whose volume tends to zero as N tends to infinity. In such circumstances one does not expect a non–trivial solution to exist [1]. In general $\widehat{\omega}_2(\alpha,\beta)$ may be infinite (Khintchine, 1926 — see [46]) - however, here, we have the restriction $\beta=\alpha^2$.

Hence it came as a surprise in 2003 when D. Roy [113] showed that the estimate (21) is optimal, by constructing examples of real (transcendental) numbers ξ for which $\widehat{\omega}_2(\xi)=\gamma+1=2.618\dots$

By means of a transference principle of Jarník (Th. 2 of [72]), Th. 1 of [115] can be reformulated as follows (see also Th. 1.5 of [116]).

Theorem 23 (Roy) *There exists a real number ξ which is neither rational nor a quadratic irrational and which has the property that, for a suitable constant $c>0$, for all sufficiently large integers N, the inequalities (22) have a solution $(x_0,x_1,x_2)\in\mathbf{Z}^2$ with $\kappa=\gamma+1$. Any such number is transcendental over $\mathbf{Q}$ and the set of such real numbers is countable.*

In [118], answering a question of Y. Bugeaud and M. Laurent [41], D. Roy shows that the exponents $\widehat{\omega}_2(\xi)$, where ξ ranges through all real numbers which are not algebraic of degree $\leqslant 2$, form a dense subset of the interval $[2,1+\gamma]$.

D. Roy calls *extremal* a number which satisfies the conditions of Theorem 23; from the point of view of approximation by quadratic polynomials, these numbers present a closest behaviour to quadratic real numbers.

Here is the first example [113] of an extremal number ξ. Recall that *the Fibonacci word*

$$w=abaababaabaababaababaabaababaabaab\dots$$

is the fixed point of the morphism $a\mapsto ab$, $b\mapsto a$. It is the limit of the sequence of words starting with $f_1=b$ and $f_2=a$ and defined inductively by concatenation as $f_n=f_{n-1}f_{n-2}$. Now let A and B be two distinct positive integers and let $\xi\in(0,1)$ be the real number whose continued fraction expansion is obtained from the Fibonacci word w by replacing the letters a and b by A and B:

$$[0;A,B,A,A,B,A,B,A,A,B,A,A,B,A,B,A,A,\dots]$$

Then ξ is extremal.

In [113, 115, 119], D. Roy investigates the approximation properties of extremal numbers by rational numbers, by quadratic numbers as well as by cubic integers.

Theorem 24 (Roy) *Let ξ be an extremal number. There exist positive constants $c_1,\dots,c_5$ with the following properties:*
(1) For any rational number $\alpha\in\mathbf{Q}$ we have

$$|\xi-\alpha|\geqslant c_1H^{-2}(\log H)^{-c_2}$$

[1] Compare with the definition of *singular systems* in § 7, Chap. V of [46].

with $H = \max\{2, \mathrm{H}(\alpha)\}$.
(2) For any algebraic number α *of degree at most* 2 *we have*

$$|\xi - \alpha| \geqslant c_3 \mathrm{H}(\alpha)^{-2\gamma-2}$$

(3) There exist infinitely many quadratic real numbers α *with*

$$|\xi - \alpha| \leqslant c_4 \mathrm{H}(\alpha)^{-2\gamma-2}$$

(4) For any algebraic integer α *of degree at most* 3 *we have*

$$|\xi - \alpha| \geqslant c_5 \mathrm{H}(\alpha)^{-\gamma-2}.$$

Moreover, in [114], he shows that for some extremal numbers ξ, property (4) holds with the exponent $-\gamma - 1$ in place of $-\gamma - 2$.

In [116] D. Roy describes the method of Davenport and Schmidt and he gives a sketch of proof of his construction of extremal numbers.

In [119] he gives a sufficient condition for an extremal number to have bounded quotients and constructs new examples of such numbers.

The values of the different exponents for the extremal numbers which are associated with Sturmian words (including the Fibonacci word) have been obtained by Y. Bugeaud and M. Laurent [41]. Furthermore, they show that the spectrum $\{\widehat{\omega}_2(\xi) \ ; \ \xi \in \mathbf{R} \setminus \overline{\mathbf{Q}}\}$ is not countable. See also their joint works [42, 43]. Their method involves words with many palindromic prefixes. S. Fischler in [61, 62] defines new exponents of approximation which allow him to obtain a characterization of the values of $\widehat{\omega}_2(\xi)$ obtained by these authors.

In [3], B. Adamczewski and Y. Bugeaud prove that for any extremal number ξ, there exists a constant $c = c(\xi)$ such that for any integer $n \geqslant 1$,

$$\omega_n(\xi) \leqslant \exp\bigl\{c\bigl(\log(3n)\bigr)^2\bigl(\log\log(3n)\bigr)^2\bigr\}.$$

In particular an extremal number is either a S–number or a T–number in Mahler's classification.

Recent results on simultaneous approximation to a number and its square, on approximation to real numbers by quadratic integers and on quadratic approximation to numbers associated with *Sturmian words* have been obtained by M. Laurent, Y. Bugeaud, S. Fischler, D. Roy...

2.4 Simultaneous rational approximation to powers of a real number

Let ξ be a real number and n a positive integer.

We consider first the simultaneous rational approximation of successive powers of ξ. We denote by $\omega'_n(\xi)$ the supremum of the real numbers w for which there exist infinitely many positive integers N for which the system

$$0 < \max_{1 \leqslant i \leqslant n} |x_i - x_0\xi^i| \leqslant N^{-w}, \quad \text{with} \quad \max_{0 \leqslant i \leqslant n} |x_i| \leqslant N, \tag{25}$$

has a solution in rational integers $x_0, x_1, \ldots, x_n$.

An upper bound for $\omega'_n(\xi)$ yields a *simultaneous approximation measure* for $\xi, \xi^2, \ldots, \xi^n$.

Next the uniform simultaneous approximation measure is the supremum $\widehat{\omega}'_n(\xi)$ of the real numbers w such that, for any sufficiently large integer N, the same system (25) has a solution in rational integers $x_0, x_1, \ldots, x_n$.

Notice that for $n = 1$, $\omega'_1(\xi) = \omega(\xi)$ and $\widehat{\omega}'_1(\xi) = \widehat{\omega}(\xi)$.

According to Dirichlet's box principle, *for all ξ and n,*

$$\frac{1}{n} \leqslant \widehat{\omega}'_n(\xi) \leqslant \omega'_n(\xi).$$

Khintchine's transference principle (see Th. B.5 in [36] and Theorem 61 below) yields relations between ω'_n and ω_n. As remarked in Theorem 2.2 of [41], the same proof yields similar relations between $\widehat{\omega}'_n$ and $\widehat{\omega}_n$.

Theorem 26 *Let n be a positive integer and ξ a real number which is not algebraic of degree $\leqslant n$. Then*

$$\frac{1}{n} \leqslant \frac{\omega_n(\xi)}{(n-1)\omega_n(\xi) + n} \leqslant \omega'_n(\xi) \leqslant \frac{\omega_n(\xi) - n + 1}{n}$$

and

$$\frac{1}{n} \leqslant \frac{\widehat{\omega}_n(\xi)}{(n-1)\widehat{\omega}_n(\xi) + n} \leqslant \widehat{\omega}'_n(\xi) \leqslant \frac{\widehat{\omega}_n(\xi) - n + 1}{n}\cdot$$

The second set of inequalities follows from the inequalities (4) and (5) of V. Jarník in Th. 3 of [72], with conditional refinements given by the inequalities (6) and (7) of the same theorem.

In particular, $\omega_n(\xi) = n$ if and only if $\omega'_n(\xi) = 1/n$. Also, $\widehat{\omega}_n(\xi) = n$ if and only if $\widehat{\omega}'_n(\xi) = 1/n$.

The spectrum of $\omega'_n(\xi)$, where ξ ranges over the set of real numbers which are not algebraic of degree $\leqslant n$, is investigated by Y. Bugeaud and M. Laurent in [42]. Only the case $n = 2$ is completely solved.

It follows from Theorem 18 that *for almost all real numbers ξ,*

$$\omega'_n(\xi) = \widehat{\omega}'_n(\xi) = \frac{1}{n} \quad \text{for all } n \geqslant 1.$$

Moreover, a consequence of Schmidt's Theorem 19 is that *for all $n \geqslant 1$ and for all algebraic real numbers ξ of degree $d > n$,*

$$\omega'_n(\xi) = \widehat{\omega}'_n(\xi) = \frac{1}{n} = \frac{1}{\omega_n(\xi)}\cdot$$

Theorems 2a and 4a of the paper [49] by H. Davenport and W.M. Schmidt (1969) imply that upper bounds for $\widehat{\omega}'_n(\xi)$ are valid for all real numbers ξ which are not algebraic of degree $\leqslant n$. For instance,

$$\widehat{\omega}'_1(\xi) = 1, \quad \widehat{\omega}'_2(\xi) \leqslant 1/\gamma = 0.618\ldots, \quad \widehat{\omega}'_3(\xi) \leqslant 1/2.$$

A slight refinement was obtained by M. Laurent [88] in 2003 (for the odd values of $n \geqslant 5$).

Theorem 27 (Davenport and Schmidt, Laurent) *Let* $\xi \in \mathbf{R} \setminus \mathbf{Q}$ *and* $n \geqslant 2$. *Assume* ξ *is not algebraic of degree* $\leqslant \lceil n/2 \rceil$. *Then*

$$\widehat{\omega}'_n(\xi) \leqslant \lceil n/2 \rceil^{-1} = \begin{cases} 2/n & \text{if } n \text{ is even,} \\ 2/(n+1) & \text{if } n \text{ is odd.} \end{cases}$$

The definition of $\widehat{\omega}'_n$ with a supremum does not reflect the accuracy of the results in [49]; for instance, the upper bound $\widehat{\omega}'_2(\xi) \leqslant 1/\gamma$ is not as sharp as Theorem 1a of [49] which is the following:

Theorem 28 (Davenport and Schmidt) *Let* ξ *be a real number which is not rational nor a quadratic irrational. There exists a constant* $c > 0$ *such that, for arbitrarily large values of* N, *the inequalities*

$$\max\{|x_1 - x_0\xi|,\ |x_2 - x_0\xi^2|\} \leqslant cN^{-1/\gamma}, \quad |x_0| \leqslant N$$

have no solution $(x_0, x_1, x_2) \in \mathbf{Z}^3$.

Before restricting ourselves to the small values of n, we emphasise that there is a huge lack in our knowledge of the spectrum of the set

$$\bigl(\omega_n(\xi),\ \widehat{\omega}_n(\xi),\ \omega'_n(\xi),\ \widehat{\omega}'_n(\xi)\bigr) \in \mathbf{R}^4,$$

where ξ ranges over the set of real numbers which are not algebraic of degree $\leqslant n$.

Consider the special case $n = 2$ and the question of quadratic approximation. As pointed out by Y. Bugeaud, a formula due to V. Jarník (1938) (Theorem 1 of [72]; see also Corollary A3 in [118] and [91]) relates $\widehat{\omega}_2$ and $\widehat{\omega}'_2$:

$$\widehat{\omega}'_2(\xi) = 1 - \frac{1}{\widehat{\omega}_2(\xi)} \cdot \tag{29}$$

Therefore the properties of $\widehat{\omega}_2$ which we considered in § 2.3 can be translated into properties of $\widehat{\omega}'_2$. For instance, $\widehat{\omega}'_2(\xi) = 1/2$ if and only if $\widehat{\omega}_2(\xi) = 2$, and this holds for almost all $\xi \in \mathbf{R}$ (see Theorem 18) and for all algebraic real numbers ξ of degree $\geqslant 3$ (see Theorem 19). If $\xi \in \mathbf{R}$ is neither rational nor a quadratic irrational, Davenport and Schmidt have proved

$$\widehat{\omega}'_2(\xi) \leqslant 1/\gamma = 0.618\ldots \tag{30}$$

The extremal numbers of D. Roy in Theorem 23 satisfy $\widehat{\omega}'_2(\xi) = 1/\gamma$. More precisely, they are exactly the numbers $\xi \in \mathbf{R}$ which are not rational nor quadratic irrationals and satisfy the following property: *there exists a constant* $c > 0$ *such that, for any sufficiently large number* N*, the inequalities*

$$\max\{|x_1 - x_0\xi|,\ |x_2 - x_0\xi^2|\} \leqslant cN^{-1/\gamma}, \quad 0 < \max\{|x_0|,\ |x_1|,\ |x_2|\} \leqslant N,$$

have a solution in rational integers x_0, x_1, x_2. (This was the original definition).

In [118], using Jarník's formula (29), D. Roy shows that the set of $(\widehat{\omega}'_2(\xi), \widehat{\omega}'_2(\xi)) \in \mathbf{R}^2$, where ξ ranges over the set of real numbers which are not algebraic of degree $\leqslant 2$, is dense in the piece of curve

$$\{(1 - t^{-1}, t)\ ;\ 2 \leqslant t \leqslant \gamma + 1\}.$$

We conclude with the case $n = 3$ and the question of cubic approximation. When $\xi \in \mathbf{R}$ is not algebraic of degree $\leqslant 3$, the estimate for $\widehat{\omega}'_3(\xi)$ by Davenport and Schmidt [49] is

$$\frac{1}{3} \leqslant \widehat{\omega}'_3(\xi) \leqslant \frac{1}{2}\cdot$$

As we have seen, the lower bound is optimal (equality holds for almost all numbers and all algebraic numbers of degree $> n$). The upper bound has been improved by D. Roy in [119]

$$\widehat{\omega}'_3(\xi) \leqslant \frac{1}{2}(2\gamma + 1 - \sqrt{4\gamma^2 + 1}) = 0.4245\ldots$$

2.5 Algebraic approximation to a single number

Let ξ be a real number and n a positive integer.

Denote by $\omega^*_n(\xi)$ the supremum of the real numbers w for which there exist infinitely many positive integers N with the following property: *there exists an algebraic number* α *of degree* $\leqslant n$ *and height* $\leqslant N$ *satisfying*[2]

$$0 < |\xi - \alpha| \leqslant N^{-w-1}. \qquad (31)$$

An upper bound for $\omega^*_n(\xi)$ is a *measure of algebraic approximation* for ξ. These numbers arise in Koksma's classification of complex numbers (Chap. III § 3 of [128] and § 3.3 of [36]).

Next, denote by $\widehat{\omega}^*_n(\xi)$ the supremum of the real numbers w such that, *for any sufficiently large integer* N*, there exists an algebraic number* α *of degree* $\leqslant n$ *and height* $\leqslant N$ *satisfying*

$$|\xi - \alpha| \leqslant \mathrm{H}(\alpha)^{-1} N^{-w}.$$

[2] The occurrence of -1 in the exponent of the right hand side of (31) is already plain for degree 1 polynomials, comparing $|\alpha - p/q|$ and $|q\alpha - p|$.

An upper bound for $\widehat{\omega}_n^*(\xi)$ yields a *uniform measure of algebraic approximation* for ξ.

From Schmidt's Subspace Theorem one deduces, for a real algebraic number ξ of degree d and for $n \geqslant 1$,

$$\widehat{\omega}_n^*(\xi) = \omega_n^*(\xi) = \min\{n, d-1\}.$$

See [36] Th. 2.9 and 2.11.

That there are relations between ω_n and ω_n^* (and, for the same reason, between $\widehat{\omega}_n$ and $\widehat{\omega}_n^*$) can be expected from Lemmas 9 and 10. Indeed, a lot of information on these numbers has been devised in order to compare the classifications of Mahler and Koksma. The estimate

$$\omega_n(\xi) \geqslant \omega_n^*(\xi),$$

which follows from Lemma 9, was known by Koksma (see also Wirsing's paper [139]). In the reversed direction, the inequalities

$$\omega_n^*(\xi) \geqslant \omega_n(\xi) - n + 1, \quad \omega_n^*(\xi) \geqslant \frac{\omega_n(\xi)+1}{2} \tag{32}$$

and

$$\omega_n^*(\xi) \geqslant \frac{\omega_n(\xi)}{\omega_n(\xi) - n + 1} \tag{33}$$

have been obtained by E. Wirsing in 1960 [139] (see § 3.4 of [36]).

A consequence is that for a real number ξ which is not algebraic of degree $\leqslant n$, if $\omega_n(\xi) = n$ then $\omega_n^*(\xi) = n$.

The inequality (33) of Wirsing has been refined in Theorem 2.1 of [41] as follows.

Theorem 34 (Bugeaud and Laurent) *Let n be a positive integer and ξ a real number which is not algebraic of degree $\leqslant n$. Then*

$$\widehat{\omega}_n^*(\xi) \geqslant \frac{\omega_n(\xi)}{\omega_n(\xi) - n + 1} \quad \text{and} \quad \omega_n^*(\xi) \geqslant \frac{\widehat{\omega}_n(\xi)}{\widehat{\omega}_n(\xi) - n + 1}.$$

A number of recent papers are devoted to this topic, including the survey given in the first part of [41] as well as Bugeaud's papers [31, 34, 5, 39, 37] where further references can be found.

We quote Proposition 2.1 of [41] which gives connections between the six exponents ω_n, $\widehat{\omega}_n$, ω_n', $\widehat{\omega}_n'$, ω_n^*, $\widehat{\omega}_n^*$.

Proposition 35 *Let n be a positive integer and ξ a real number which is not algebraic of degree $\leqslant n$. Then*

$$\frac{1}{n} \leqslant \widehat{\omega}_n'(\xi) \leqslant \min\{1, \omega_n'(\xi)\}$$

and

$$1 \leqslant \widehat{\omega}_n^*(\xi) \leqslant \min\bigl\{\omega_n^*(\xi),\ \widehat{\omega}_n(\xi)\bigr\} \leqslant \max\bigl\{\omega_n^*(\xi),\ \widehat{\omega}_n(\xi)\bigr\} \leqslant \omega_n(\xi).$$

A further relation connecting ω_n^* and $\widehat{\omega}'_n$ has been discovered by H. Davenport and W.M. Schmidt in 1969 [49]. We discuss their contribution in § 2.6. For our immediate concern here we only quote the following result:

Theorem 36 *Let n be a positive integer and ξ a real number which is not algebraic of degree $\leqslant n$, Then*

$$\omega_n^*(\xi)\widehat{\omega}'_n(\xi) \geqslant 1.$$

The spectral question for ω_n^* is one of the main challenges in this domain. Wirsing's conjecture states that for any integer $n \geqslant 1$ and any real number ξ which is not algebraic of degree $\leqslant n$, we have $\omega_n^*(\xi) \geqslant n$. In other terms:

Conjecture 37 (Wirsing) *For any $\epsilon > 0$ there is a constant $c(\xi, n, \epsilon) > 0$ for which there are infinitely many algebraic numbers α of degree $\leqslant n$ with*

$$|\xi - \alpha| \leqslant c(\xi, n, \epsilon)\mathrm{H}(\alpha)^{-n-1+\epsilon}.$$

In 1960, E. Wirsing [139] proved that for any real number which is not algebraic of degree $\leqslant n$, the lower bound $\omega_n^*(\xi) \geqslant (n+1)/2$ holds: it suffices to combine (32) with the lower bound $\omega_n(\xi) \geqslant n$ from (17) (see [124] Chap. VIII Th. 3B). More precisely, he proved that for such a $\xi \in \mathbf{R}$ *there is a constant $c(\xi, n) > 0$ for which there exist infinitely many algebraic numbers α of degree $\leqslant n$ with*

$$|\xi - \alpha| \leqslant c(\xi, n)\mathrm{H}(\alpha)^{-(n+3)/2}.$$

The special case $n = 2$ of this estimate was improved in 1967 when H. Davenport and W.M. Schmidt [48] replaced $(n + 3)/2 = 5/2$ by 3. This is optimal for the approximation to a real number by quadratic algebraic numbers. This is the only case where Wirsing's Conjecture is solved. More recent estimates are due to V.I. Bernik and K. Tishchenko [28, 132, 133, 134, 135, 136]. This question is studied by Y. Bugeaud in his book [36] (§ 3.4) where he proposes the following *Main Problem*:

Conjecture 38 (Bugeaud) *Let $(w_n)_{n\geqslant 1}$ and $(w_n^*)_{n\geqslant 1}$ be two non–decreasing sequences in $[1, +\infty]$ for which*

$$n \leqslant w_n^* \leqslant w_n \leqslant w_n^* + n - 1 \quad \textit{for any } n \geqslant 1.$$

Then there exists a transcendental real number ξ for which

$$\omega_n(\xi) = w_n \quad \textit{and} \quad \omega_n^*(\xi) = w_n^* \quad \textit{for any } n \geqslant 1.$$

A summary of known results on this problem is given in § 7.8 of [36].

The spectrum

$$\bigl\{\omega_n(\xi) - \omega_n^*(\xi)\ ;\ \ \xi \in \mathbf{R} \text{ not algebraic of degree} \leqslant n\bigr\} \subset [0, n - 1]$$

of $\omega_n - \omega_n^*$ for $n \geqslant 2$ was studied by R.C. Baker in 1976 who showed that it contains $[0, 1-(1/n)]$. This has been improved by Y. Bugeaud in [34]: it contains the interval $[0, n/4]$.

Most results concerning $\omega_n^*(\xi)$ and $\widehat{\omega}_n^*(\xi)$ for $\xi \in \mathbf{R}$ have extensions to complex numbers, only the numerical estimates are slightly different. However see [40].

2.6 Approximation by algebraic integers

An innovative and powerful approach was initiated in the seminal paper [49] by H. Davenport and W.M. Schmidt (1969). It rests on the transference principle arising from the geometry of numbers and Mahler's theory of *polar convex bodies* and allows to deal with approximation by algebraic integers of bounded degree. The next statement includes a refinement by Y. Bugeaud and O. Teulié (2000) [45] who observed that one may treat approximations by algebraic integers of given degree; the sharpest results in this direction are due to M. Laurent [88].

From the estimate $\omega_n^*(\xi)\widehat{\omega}_n'(\xi) \geqslant 1$ in Theorem 36 one deduces the following statement. Let n be a positive integer and let ξ be a real number which is not algebraic of degree $\leqslant n$. Let λ satisfy $\widehat{\omega}_n'(\xi) < \lambda$. Then for $\kappa = (1/\lambda) + 1$, there is a constant $c(n, \xi, \kappa) > 0$ such that the equation

$$|\xi - \alpha| \leqslant c(n, \xi, \kappa)\mathrm{H}(\alpha)^{-\kappa} \tag{39}$$

has infinitely many solutions in algebraic numbers α of degree n. In this statement one may replace "algebraic numbers α of degree n" by "algebraic integers α of degree $n+1$" and also by "algebraic units α of degree $n+2$".

Proposition 40 *Let $\kappa > 1$ be a real number, n be a positive integer and ξ be a real number which is not algebraic of degree $\leqslant n$. Assume $\widehat{\omega}_n'(\xi) < 1/(\kappa - 1)$. Then there exists a constant $c(n, \xi, \kappa) > 0$ such that there are infinitely many algebraic integers α of degree $n+1$ satisfying (39) and there are infinitely many algebraic units α of degree $n+2$ satisfying (39).*

Suitable values for κ are deduced from Theorem 27 and estimate (21). For instance, from Theorem 36 and the estimate $\widehat{\omega}_2'(\xi) \leqslant 1/\gamma$ of Davenport and Schmidt in (30) one deduces $\omega_2^*(\xi) \geqslant \gamma$. Hence for any $\kappa < 1 + \gamma$ the the assumptions of Proposition 40 are satisfied. More precisely, the duality (or transference) arguments used by Davenport and Schmidt to prove Theorem 36 together with their Theorem 28 enabled them to deduce the next statement ([49], Th. 1).

Theorem 41 (Davenport and Schmidt) *Let $\xi \in \mathbf{R}$ be a real number which is neither rational nor a quadratic irrational. Then there is a constant $c > 0$ with the following property: there are infinitely many algebraic integers α of degree at most 3 which satisfy*

$$0 < |\xi - \alpha| \leqslant c\mathrm{H}(\alpha)^{-\gamma-1}. \tag{42}$$

Lemma 9 shows that under the same assumptions, for another constant $c > 0$ there are infinitely many monic polynomials $P \in \mathbf{Z}[X]$ of degree at most 3 satisfying

$$|P(\xi)| \leqslant c\mathrm{H}(P)^{-\gamma}. \tag{43}$$

Estimates (42) and (43) are optimal for certain classes of *extremal* numbers [114]. Approximation of extremal numbers by cubic integers are studied by D. Roy in [114, 115]. Further papers dealing with approximation by algebraic integers include [45, 135, 122, 4].

Another development of the general and powerful method of Davenport and Schmidt deals with the question of approximating simultaneously several numbers by conjugate algebraic numbers: this is done in [122] and refined in [117] by D. Roy. Also in [117] D. Roy gives variants of Gel'fond's Transcendence Criterion involving not only a single number ξ but sets $\{\gamma+\xi_1,\dots,\gamma+\xi_m\}$ or $\{\gamma\xi_1,\dots,\gamma\xi_m\}$. In two recent manuscripts [121, 120], D. Roy produces new criteria for the additive and for the multiplicative groups.

A different application of transference theorems is to link inhomogeneous Diophantine approximation problems with homogeneous ones [42].

2.7 Overview of metrical results for polynomials

Here we give a brief account of some significant results that have produced new ideas and generalisations, as well as some interesting problems and conjectures. We begin with the probabilistic theory (that is, Lebesgue measure statements) and continue with the more delicate Hausdorff measure/dimension results. Results for multivariable polynomials, in particular, the recent proof of a conjecture of Nesterenko on the measure of algebraic independence of almost all real m-tuples, will be sketched in § 3.6, as will metrical results on simultaneous approximation. Note that many of the results suggested here have been established in the far more general situation of Diophantine approximation on manifolds. However, for simplicity, we will only explain this Diophantine approximation for the case of integral polynomials.

Mahler's problem [96], which arose from his classification of real (and complex) numbers, remains a major influence over the metrical theory of Diophantine approximation. As mentioned in §2.3, the problem has been settled by Sprindzuk in 1965. Answering a question posed by A. Baker in [9], Bernik [25] established a generalisation of Mahler's problem akin to Khintchine's one-dimensional convergence result in Theorem 7, involving the critical sum

$$\sum_{h=1}^{\infty} \Psi(h) \tag{44}$$

of values of the function $\Psi : \mathbb{N} \to \mathbb{R}^+$ that defines the error of approximation.

Theorem 45 (Bernik, 1989) *Given a monotonic Ψ such that the critical sum (44) converges, for almost all $\xi \in \mathbb{R}$ the inequality*

$$|P(\xi)| < H(P)^{-n+1}\Psi(H(P)) \tag{46}$$

has only finitely many solutions in $P \in \mathbb{Z}[x]$ with $\deg P \leqslant n$.

In the case $n = 1$, inequality (46) reduces to rational approximations of real numbers and is covered by Khintchine's theorem 7 [73]. Khintchine's theorem 7 also covers the solubility of (46) in case when $n = 1$ and (44) diverges. For arbitrary n the complementary divergence case of Theorem 45 has been established by Beresnevich, who has shown in [13] that if (44) diverges then for almost all real ξ inequality (46) has infinitely many solutions $P \in \mathbb{Z}[x]$ with $\deg P = n$. In fact the latter statement follows from the following analogue of Khintchine's theorem 7 for approximation by algebraic numbers, also established in [13].

Theorem 47 (Beresnevich, 1999) *Let $n \in \mathbb{N}$, $\Psi : \mathbb{N} \to \mathbb{R}^+$ be a monotonic error function and $\mathcal{A}_n(\Psi)$ be the set of real ξ such that*

$$|\xi - \alpha| < H(\alpha)^{-n}\Psi(H(\alpha)) \tag{48}$$

has infinitely many solutions in real algebraic numbers of degree $\deg \alpha = n$. Then $\mathcal{A}_n(\Psi)$ has full Lebesgue measure if the sum (44) diverges and zero Lebesgue measure otherwise.

Bugeaud [32] has proved an analogue of Theorem 47 for approximation by algebraic integers:

Theorem 49 (Bugeaud, 2002) *Let $n \in \mathbb{N}$, $n \geqslant 2$, $\Psi : \mathbb{N} \to \mathbb{R}^+$ be a monotonic error function and $\mathcal{I}_n(\Psi)$ be the set of real ξ such that*

$$|\xi - \alpha| < H(\alpha)^{-n+1}\Psi(H(\alpha)) \tag{50}$$

has infinitely many solutions in real algebraic integers of degree $\deg \alpha = n$. Then $\mathcal{I}_n(\Psi)$ has full Lebesgue measure if the sum (44) diverges and zero Lebesgue measure otherwise.

No analogue of Theorem 45 is known for the monic polynomial case — however see the final section of [23].

Unlike Theorem 45, the convergence parts of Theorems 47 and 49 are rather trivial consequences of the Borel-Cantelli Lemma, which also implies that the monotonicity condition is unnecessary in the case of convergence. The intriguing question now arises whether the monotonicity condition in Theorem 45 and in the divergence part of Theorems 47 and 49 can be dropped. Beresnevich [16] has recently shown that the monotonicity condition on Ψ can indeed be safely removed from Theorem 45. Regarding Theorems 47 and 49 removing the monotonicity condition is a fully open problem – [16]. In fact, in dimension $n = 1$ removing the monotonicity from Theorem 47 falls within the Duffin and Schaeffer problem [56]. Note, however, that the higher dimensional Duffin-Schaeffer problem has been settled in the affirmative [108].

It is interesting to compare Theorems 47 and 49 with their global counterparts. In the case of approximation by algebraic numbers of degree $\leqslant n$, the appropriate statement is known as the Wirsing conjecture 37 (see § 2.5). The latter has been verified for $n = 2$ by Davenport and Schmidt but is open in higher dimensions. Theorem 47 implies that the statement of the Wirsing conjecture 37 holds for almost all real ξ – the actual conjecture states that it is true at least for all transcendental ξ. In the case of approximation by algebraic integers of degree $\leqslant n$, Roy has shown that the statement analogous to Wirsing's conjecture is false [114]. However, Theorem 49 implies that the statement holds for almost all real ξ. In line with the recent 'metrical' progress on Littlewood's conjecture by Einsiedler, Katok and Lindenstrauss [57], it would be interesting to find out whether the set of possible exceptions to the Wirsing-Schmidt conjecture is of Hausdorff dimension zero. A similar question can also be asked about approximation by algebraic integers; this would shed light on the size of the set of exceptions, shown to be non–empty by Roy.

A. Baker [10] suggested a strengthening of Mahler's problem in which the height $H(P) = \max\{|a_n|, \dots, |a_0|\}$ of polynomial $P(x) = a_n x^n + \dots + a_1 x + a_0$ is replaced by

$$H^\times(P) = \prod_{i=1}^{n} \max\{1, |a_i|\}^{1/n}.$$

The corresponding statement has been established by Kleinbock and Margulis [76] in a more general context of Diophantine approximation on manifolds. Specialising their result to polynomials gives the following

Theorem 51 (Kleinbock & Margulis, 1998) *Let $\varepsilon > 0$. Then for almost all $\xi \in \mathbb{R}$ the inequality*

$$|P(\xi)| < H^\times(P)^{-n-\varepsilon} \tag{52}$$

has only finitely many solutions in $P \in \mathbb{Z}[x]$ with $\deg P \leqslant n$.

A multiplicative analogue of Theorem 45 with $H(P)$ replaced by $H^\times(P)$ has been obtained by Bernik, Kleinbock and Margulis [27] (also within the framework of manifolds). Note that in their theorem the convergence of (44) must be replaced by the stronger condition that $\sum_{h=1}^{\infty} \Psi(h)(\log h)^{n-1} < \infty$. This condition is believed to be optimal but it is not known if the multiplicative analogue of Theorem 47, when $H(P)$ is replaced by $H^\times(P)$, holds. In [23], Beresnevich and Velani have proved an inhomogeneous version of the theorem of Kleinbock and Margulis; and, in particular, an inhomogeneous version of Theorem 51.

With [11], A. Baker and W.M. Schmidt pioneered the use of Hausdorff dimension in the context of approximation of real numbers by algebraic numbers with a natural generalisation of the Jarník–Besicovitch theorem:

Theorem 53 (Baker & Schmidt, 1970) *Let* $w \geqslant n$. *Then the set of* $\xi \in \mathbb{R}$ *for which*

$$|\xi - \alpha| < H(\alpha)^{-w} \tag{54}$$

holds for infinitely many algebraic numbers α *with* $\deg \alpha \leqslant n$ *has Hausdorff dimension* $(n+1)/(w+1)$.

In particular, Theorem 53 implies that the set

$$A(w) = \Big\{\xi \in \mathbb{R} : |P(\xi)| < H(P)^{-w} \text{ for infinitely many } P \in \mathbb{Z}[x],\ \deg P \leqslant n\Big\} \tag{55}$$

has Hausdorff dimension at least $(n+1)/(w+1)$. Baker and Schmidt conjectured that this lower bound is sharp, and this was established by Bernik in [24]:

Theorem 56 (Bernik, 1983) *Let* $w \geqslant n$. *Then* $\dim A(w) = \dfrac{n+1}{w+1}$.

This theorem has an important consequence for the spectrum of Diophantine exponents already discussed (see Chap. 5 of [36]). Bugeaud [32] has obtained an analogue of Theorem 53 in the case of algebraic integers. However, obtaining an analogue of Theorem 56 for the case of algebraic integers is as yet an open problem.

Recently, Beresnevich, Dickinson and Velani have established a sharp Hausdorff measure version of Theorem 53, akin to a classical result of Jarník. In order to avoid introducing various related technicalities, we refer the reader to [21, § 12.2]. Their result implies the corresponding divergent statement for the Hausdorff measure of the set of $\xi \in \mathbb{R}$ such that (46) holds infinitely often. Obtaining the corresponding convergent statement represents yet another open problem.

There are various generalisations of the above results to the case of complex and p-adic numbers and more generally to the case of S-arithmetic (for instance by D. Kleinbock and G. Tomanov in [77]).

3 Simultaneous Diophantine approximation in higher dimensions

In § 2.3, we considered polynomial approximation to a complex number ξ, which is the study of $|P(\xi)|$ for $P \in \mathbf{Z}[X]$. As we have seen, negative results on the existence of polynomial approximations lead to *transcendence measures.* A more general situation is to fix several complex numbers $x_1, \dots, x_m$ and to study the smallness of polynomials in these numbers – negative results provide *measures of algebraic independence* to $x_1, \ldots, x_m$.

This is a again a special case, where $\xi_i = x_1^{a_1} \cdots x_m^{a_m}$, of the study of linear combinations in $\xi_1, \ldots, \xi_n$, where $\xi_1, \ldots, \xi_n$ are given complex numbers. Now, negative results are *measures of linear independence* to $\xi_1, \ldots, \xi_n$.

There are still more general situations which we are not going to consider thoroughly but which are worth mentioning, namely the study of *simultaneous approximation of dependent quantities* and *approximation on a manifold* (see for instance [26]).

We start with the question of algebraic independence (§ 3.1) in connection with extensions to higher dimensions of Gel'fond's Criterion 11. Next (§§ 3.2 and 3.3) we discuss a recent work by M. Laurent [91], who introduces further coefficients for the study of simultaneous approximation. The special case of two numbers (§ 3.4) is best understood so far.

There is a very recent common generalisation of the question of Diophantine approximation to a point in $\mathbf{R}^n$ which is considered in § 3.3 on the one hand, and of the question of approximation to a real number by algebraic numbers of bounded degree considered in § 2.5 on the other hand. It consists in the investigation of the approximation to a point in $\mathbf{R}^n$ by algebraic hypersurfaces, or more generally algebraic varieties defined over the rationals. This topic has been recently investigated by W.M. Schmidt in [127] and [126].

3.1 Criteria for algebraic independence

In § 2.2 we quoted Gel'fond's algebraic independence results of two numbers of the form α^{β^i} $(1 \leqslant i \leqslant d-1)$. His method has been extended after the work of several mathematicians including A.O. Gel'fond, A.A. Smelev, W.D. Brownawell, G.V. Chudnovskii, P. Philippon, Yu.V. Nesterenko, G. Diaz (see [68], [79], [131], [101, 102], [104, 107], [59] Chap. 6 and [103]). So far the best known result, due to G. Diaz [50], proves "half" of what is expected.

Theorem 57 *Let β be an algebraic number of degree $d \geqslant 2$ and α a non–zero algebraic number. Moreover, let $\log \alpha$ be any non–zero logarithm of α. Write α^z in place of $\exp(z \log \alpha)$. Then among the numbers*

$$\alpha^{\beta},\ \alpha^{\beta^2},\ \ldots,\alpha^{\beta^{d-1}},$$

at least $\lceil (d+1)/2 \rceil$ are algebraically independent.

In order to prove such a result, as pointed out by S. Lang in [78], it would have been sufficient to replace the Transcendence Criterion Theorem 11 by a criterion for algebraic independence. However, an example, going back to A.Ya. Khintchine in 1926 [74] and quoted in J.W.S. Cassels's book ([46] Chap. V, Th. 14; see also the appendix of [104] and appendix A of [121]), shows that in higher dimensions, some extra hypothesis cannot be avoided (and this is a source of difficulty in the proof of Theorem 57). After the work of W.D. Brownawell and G.V. Čudnovs′kiĭ, such criteria were proved by P. Philippon [104, 107], Yu.V. Nesterenko [101, 102], M. Ably, C. Jadot (further references are given in [59] and [138] § 15.5). Reference [103] is an introduction to algebraic independence theory which includes a chapter on multihomogeneous elimination by G. Rémond [109] and a discussion of criteria for algebraic independence by P. Philippon [107].

Further progress has been made by M. Laurent and D. Roy in 1999 who produced criteria with multiplicities [92, 93] (see also [94]) and considered questions of approximation by algebraic sets. Moreover, in [93] they investigate the approximation properties, by algebraic numbers of bounded degree and height, of a m-tuple which generates a field of transcendence degree 1 – this means that the corresponding point in $\mathbf{C}^m$ belongs to an affine curve defined over $\mathbf{Q}$. For $m = 1$ they proved in [92] the existence of approximation; this has been improved by G. Diaz in [51]. Further contributions are due to P. Philippon (see for instance [106]).

A very special case of the investigation of Laurent and Roy is a result related to Wirsing's lower bound for ω_n^* (see § 2.5), with a weak numerical constant, but with a lower bound for the degree of the approximation. Their result (Corollary 1 of § 2 of [93]) has been improved by Y. Bugeaud and O. Teulié (Corollary 5 of [45]) who prove that the approximations α can be required to be algebraic numbers of exact degree n or algebraic integers of exact degree $n + 1$.

Theorem 58 (Bugeaud and Teulié) *Let $\epsilon > 0$ be a real number, $n \geqslant 2$ be an integer and ξ a real number which is not algebraic of degree n. Then the inequality*

$$|\xi - \alpha| \leqslant \mathrm{H}(\alpha)^{-((n+3)/2)+\epsilon}$$

has infinitely many solutions in algebraic integers α of degree n.

The ϵ in the exponent can be removed by introducing a constant factor. Further, for almost all $\alpha \in \mathbf{R}$, the result holds with the exponent replaced by n, as shown by Y. Bugeaud in [33].

There are close connections between questions of algebraic independence and simultaneous approximation of numbers. We shall not discuss this subject thoroughly here; it would deserve another survey. We just quote a few recent papers.

Applications of Diophantine approximation questions to transcendental number theory are considered by P. Philippon in [105]. M. Laurent [87] gives heuristic motivations in any transcendence degree. Conjecture 15.31 of [138] on simultaneous approximation of complex numbers suggests a path towards results on large transcendence degree.

In [110] D. Roy shows some limitations of the conjectures of algebraic approximation by constructing points in $\mathbf{C}^m$ which do not have good algebraic approximations of bounded degree and height, when the bounds on the degree and height are taken from specific sequences. The coordinates of these points are Liouville numbers.

3.2 Four exponents: asymptotic or uniform simultaneous approximation by linear forms or by rational numbers

Let $\xi_1, \ldots, \xi_n$ be real numbers. Assume that the numbers $1, \xi_1, \ldots, \xi_n$ are linearly independent over $\mathbf{Q}$. There are (at least) two points of view for studying approximation to $\xi_1, \ldots, \xi_n$. On the one hand, one may consider linear forms (see for instance [78])

$$|x_0 + x_1\xi_1 + \cdots + x_n\xi_n|.$$

On the other hand, one may investigate the existence of simultaneous approximation by rational numbers

$$\max_{1\leqslant i\leqslant n} \left|\xi_i - \frac{x_i}{x_0}\right|.$$

Each of these two points of view has two versions, an asymptotic one (with exponent denoted ω) and a uniform one (with exponent denoted $\widehat{\omega}$). This gives rise to four exponents introduced in [42] (see also [91]),

$$\omega(\theta), \quad \widehat{\omega}(\theta), \quad \omega({}^t\theta), \quad \widehat{\omega}({}^t\theta),$$

where

$$\theta = (\xi_1, \ldots, \xi_n) \quad \text{and} \quad {}^t\theta = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}.$$

We shall recover the situation of §§ 2.1 and 2.4 in the special case where $\xi_i = \xi^i$, $1 \leqslant i \leqslant n$: for $\theta = (\xi, \xi^2, \ldots, \xi^n)$,

$$\omega(\theta) = \omega_n(\xi), \quad \widehat{\omega}(\theta) = \widehat{\omega}_n(\xi), \quad \omega({}^t\theta) = \omega'_n(\xi), \quad \widehat{\omega}({}^t\theta) = \widehat{\omega}'_n(\xi).$$

Notice that the index n is implicit in the notations involving ω, since it is the number of components of θ.

We start with the question of *asymptotic approximation by linear forms*. We denote by $\omega(\theta)$ the supremum of the real numbers w for which there exist infinitely many positive integers N for which the system

$$|x_0 + x_1\xi_1 + \cdots + x_n\xi_n| \leqslant N^{-w}, \quad 0 < \max_{0\leqslant i\leqslant n} |x_i| \leqslant N, \tag{59}$$

has a solution in rational integers $x_0, x_1, \ldots, x_n$. An upper bound for $\omega(\theta)$ is a *linear independence measure* for $1, \xi_1, \ldots, \xi_n$.

The hat version of $\omega(\theta)$ is, as expected, related to the study of *uniform approximation by linear forms*: we denote by $\widehat{\omega}(\theta)$ the supremum of the real numbers w such that, for any sufficiently large integer N, the same system (59) has a solution.

Obviously $\widehat{\omega}(\theta) \leqslant \omega(\theta)$.

The second question is that of asymptotic simultaneous approximation by rational numbers. Following again [91], we denote by $\omega({}^t\theta)$ the supremum of the real numbers w for which there exist infinitely many positive integers N for which the system

$$\max_{1\leqslant i\leqslant n} |x_i - x_0\xi_i| \leqslant N^{-w}, \quad \text{with} \quad 0 < \max_{0\leqslant i\leqslant n} |x_i| \leqslant N \tag{60}$$

has a solution in rational integers $x_0, x_1, \ldots, x_n$. An upper bound for $\omega({}^t\theta)$ is a *simultaneous approximation measure* for $1, \xi_1, \ldots, \xi_n$.

The uniform simultaneous approximation by rational numbers is measured by the hat version of ω: we denote by $\widehat{\omega}({}^t\theta)$ the supremum of the real numbers w such that, for any sufficiently large integer N, the same system (60) has a solution.

Again $\widehat{\omega}({}^t\theta) \leqslant \omega({}^t\theta)$.

Transference principles provide relations between $\omega(\theta)$ and $\omega({}^t\theta)$. The next result (Khintchine, 1929 [74]) shows that $\omega(\theta) = n$ if and only if $\omega({}^t\theta) = 1/n$.

Theorem 61 (Khintchine transference principle) *If we set* $\omega = \omega(\theta)$ *and* ${}^t\omega = \omega({}^t\theta)$*, then we have*

$$\omega \geqslant n\,{}^t\omega + n - 1 \quad \text{and} \quad {}^t\omega \geqslant \frac{\omega}{(n-1)\omega + n}\cdot$$

In order to study these numbers, M. Laurent introduces in [91] further exponents as follows.

3.3 Further exponents, following M. Laurent

For each d in the range $0 \leqslant d \leqslant n-1$, M. Laurent [91] introduces two exponents, one for asymptotic approximation $\omega_d(\theta)$ and one for uniform approximation $\widehat{\omega}_d(\theta)$, which measures the quality of simultaneous approximation to the given tuple $\theta = (\xi_1, \ldots, \xi_n)$ from various points of view. First embed $\mathbf{R}^n$ into $\mathbf{P}^n(\mathbf{R})$ by mapping $\theta = (\xi_1, \ldots, \xi_n)$ to $(\xi_1 : \cdots : \xi_n : 1)$.

Now for $0 \leqslant d \leqslant n-1$, define

$$\omega_d(\theta) = \sup\Big\{w;\ \text{there exist infinitely many vectors}$$
$$X = x_0 \wedge \cdots \wedge x_d \in \Lambda^{d+1}(\mathbf{Z}^{n+1}) \ \text{ for which } |X \wedge \theta| \leqslant |X|^{-w}\Big\}$$

and

$$\widehat{\omega}_d(\theta) = \sup\Big\{w;\ \text{for any sufficiently large } N, \text{ there exists}$$
$$X = x_0 \wedge \cdots \wedge x_d \in \Lambda^{d+1}(\mathbf{Z}^{n+1}) \text{ such that } 0 < |X| \leqslant N \quad \text{and} \quad |X \wedge \theta| \leqslant N^{-w}\Big\}.$$

Hence $\omega_d(\theta) \geqslant \widehat{\omega}_d(\theta)$.

The multivector $X = x_0 \wedge \cdots \wedge x_d$ is a system of Plücker coordinates of the linear projective subvariety $L = \langle x_0, \ldots, x_d \rangle \subset \mathbf{P}^n(\mathbf{R})$. Then

$$\frac{|X \wedge \theta|}{|X||\theta|}$$

is essentially the distance $d(\theta, L) = \min_{x \in L} d(\theta, x)$ between the image of θ in $\mathbf{P}^n(\mathbf{R})$ to L. As a consequence, equivalent definitions are as follows, where $\mathrm{H}(L)$ denotes the Weil height of any system of Plücker coordinates of L.

$$\omega_d(\theta) = \sup \Big\{ w;\ \text{there exist infinitely many } L, \text{ rational over } \mathbf{Q}, \\ \dim L = d \ \text{ and } \ d(\theta, L) \leqslant \mathrm{H}(L)^{-w-1} \Big\}$$

and

$$\widehat{\omega}_d(\theta) = \sup \Big\{ w;\ \text{for any sufficiently large } N, \text{ there exists } L, \text{ rational over } \mathbf{Q}, \\ \dim L = d,\ \mathrm{H}(L) \leqslant N \text{ and } d(\theta, L) \leqslant \mathrm{H}(L)^{-1} N^{-w} \Big\}.$$

In the extremal cases $d = 0$ and $d = n - 1$, one recovers the exponents of § 3.2:

$$\omega_0(\theta) = \omega({}^t\theta), \quad \widehat{\omega}_0(\theta) = \widehat{\omega}({}^t\theta), \quad \omega_{n-1}(\theta) = \omega(\theta), \qquad \widehat{\omega}_{n-1}(\theta) = \widehat{\omega}(\theta).$$

The lower bound

$$\widehat{\omega}_d(\theta) \geqslant \frac{d+1}{n-d} \quad \text{for all} \quad d = 0, \ldots, n-1$$

valid for all θ (with $1, \xi_1, \ldots, \xi_n$ linearly independent over $\mathbf{Q}$) follows from the results of W.M. Schmidt in his foundational paper [123] (see [44]). In particular for $d = n - 1$ and $d = 0$ respectively, this lower bound yields

$$\widehat{\omega}(\theta) \geqslant n \quad \text{and} \quad \widehat{\omega}({}^t\theta) \geqslant 1/n$$

and in the special case $\xi_i = \xi^i$ $(1 \leqslant i \leqslant n)$ one recovers the lower bounds

$$\widehat{\omega}_n(\xi) \geqslant n \quad \text{and} \quad \widehat{\omega}'_n(\xi) \geqslant 1/n,$$

which we deduced in § 2.3 and § 2.4 respectively from Dirichlet's box principle.

It was proved by Khintchine in 1926 [74] that $\omega(\theta) = n$ if and only if $\omega({}^t\theta) = 1/n$. In [91], M. Laurent slightly improves on earlier inequalities due to W.M. Schmidt [123] splitting the classical Khintchine's transference principle (Theorem 61) into intermediate steps.

Theorem 62 (Schmidt, Laurent) *Fix $n \geqslant 1$ and $\theta = (\xi_1, \ldots, \xi_n) \in \mathbf{R}^n$. Set $\omega_d = \omega_d(\theta)$, $0 \leqslant d \leqslant n-1$. The "going up transference principle" is*

$$\omega_{d+1} \geqslant \frac{(n-d)\omega_d + 1}{n-d-1}, \quad 0 \leqslant d \leqslant n-2,$$

while the "going down transference principle" is

$$\omega_{d-1} \geqslant \frac{d\omega_d}{\omega_d + d + 1}, \quad 1 \leqslant d \leqslant n-1.$$

Moreover these estimates are optimal.

As a consequence of Theorem 62, one deduces that if $\omega_d = (d+1)/(n-d)$ for one value of d in the range $0 \leqslant d \leqslant n-1$, then the same equality holds for all $d = 0, 1, \dots, n-1$. Hence, for almost all $\theta \in \mathbf{R}^n$,

$$\omega_d(\theta) = \widehat{\omega}_d(\theta) = \frac{d+1}{n-d} \quad \text{for} \quad 0 \leqslant d \leqslant n-1.$$

A complement to Theorem 62, involving the hat coefficients, is given in [91] Th. 3.

A problem raised in [91] is to *find the spectrum in* $(\mathbf{R} \cup \{+\infty\})^n$ *of the* n*-tuples*

$$\big(\omega_0(\theta), \dots, \omega_{n-1}(\theta)\big),$$

where θ *ranges over the elements* $(\xi_1, \dots, \xi_n)$ *in* $\mathbf{R}^n$ *with* $1, \xi_1, \dots, \xi_n$ *linearly independent over* $\mathbf{Q}$. Partial results are given in [91].

In [42] Y. Bugeaud and M. Laurent define and study exponents of *inhomogeneous* Diophantine approximation. Further progress on this topic has been achieved by M. Laurent in [89].

3.4 Dimension 2

We consider the special case $n = 2$ of § 3.3: we replace (ξ_1, ξ_2) by (ξ, η). So let ξ and η be two real numbers with $1, \xi, \eta$ linearly independent over $\mathbf{Q}$.

Khintchine's transference Theorem 61 reads in this special case

$$\frac{\omega(\xi,\eta)}{\omega(\xi,\eta)+2} \leqslant \omega\begin{pmatrix}\xi\\ \eta\end{pmatrix} \leqslant \frac{\omega(\xi,\eta)-1}{2}\cdot$$

V. Jarník studied these numbers in a series of papers from 1938 to 1959 (see [36, 42, 90]). He proved that both sides are optimal. Also Jarník's formula (of which (29) is a special case) reads:

$$\widehat{\omega}\begin{pmatrix}\xi\\ \eta\end{pmatrix} = 1 - \frac{1}{\widehat{\omega}(\xi,\eta)}\cdot \tag{63}$$

The spectrum of each of our four exponents is as follows:

$\omega(\xi,\eta)$ takes any value in the range $[2, +\infty]$,

$\omega\begin{pmatrix}\xi\\ \eta\end{pmatrix}$ takes any value in the range $[1/2, 1]$,

$\widehat{\omega}(\xi, \eta)$ takes any value in the range $[2, +\infty]$,
$\widehat{\omega}\begin{pmatrix}\xi\\ \eta\end{pmatrix}$ takes any value in the range $[1/2, 1]$.

Moreover, for almost all $(\xi, \eta) \in \mathbf{R}^2$,

$$\omega(\xi, \eta) = \widehat{\omega}(\xi, \eta) = 2, \quad \omega\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \widehat{\omega}\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \frac{1}{2}\cdot$$

A more precise description of the spectrum of the quadruple is due to M. Laurent [90]:

Theorem 64 (Laurent) *Assume $1, \xi, \eta$ are linearly independent over $\mathbf{Q}$. The four exponents*

$$\omega = \omega(\xi, \eta), \quad \omega' = \omega\begin{pmatrix}\xi\\ \eta\end{pmatrix}, \quad \widehat{\omega} = \widehat{\omega}(\xi, \eta), \quad \widehat{\omega}' = \widehat{\omega}\begin{pmatrix}\xi\\ \eta\end{pmatrix}$$

are related by

$$2 \leqslant \widehat{\omega} \leqslant +\infty, \quad \widehat{\omega}' = \frac{\widehat{\omega} - 1}{\widehat{\omega}}, \quad \frac{\omega(\widehat{\omega} - 1)}{\omega + \widehat{\omega}} \leqslant \omega' \leqslant \frac{\omega - \widehat{\omega} + 1}{\widehat{\omega}}$$

with the obvious interpretation if $\omega = +\infty$. Conversely, for any $(\omega, \omega', \widehat{\omega}, \widehat{\omega}')$ in $(\mathbf{R}_{>0} \cup \{+\infty\})^4$ satisfying the previous inequalities, there exists $(\xi, \eta) \in \mathbf{R}^2$, with $1, \xi, \eta$ linearly independent over $\mathbf{Q}$, for which

$$\omega = \omega(\xi, \eta), \quad \omega' = \omega\begin{pmatrix}\xi\\ \eta\end{pmatrix}, \quad \widehat{\omega} = \widehat{\omega}(\xi, \eta), \quad \widehat{\omega}' = \widehat{\omega}\begin{pmatrix}\xi\\ \eta\end{pmatrix}.$$

As a consequence:

Corollary 65 *The exponents $\omega = \omega(\xi, \eta)$, $\widehat{\omega} = \widehat{\omega}(\xi, \eta)$ are related by*

$$\omega \geqslant \widehat{\omega}(\widehat{\omega} - 1) \quad \textit{and} \quad \widehat{\omega} \geqslant 2.$$

Conversely, for any $(\omega, \widehat{\omega})$ satisfying these conditions, there exists (ξ, η) for which

$$\omega(\xi, \eta) = \omega \quad \textit{and} \quad \widehat{\omega}(\xi, \eta) = \widehat{\omega}.$$

Corollary 66 *The exponents $\omega' = \omega\begin{pmatrix}\xi\\ \eta\end{pmatrix}$, $\widehat{\omega}' = \widehat{\omega}\begin{pmatrix}\xi\\ \eta\end{pmatrix}$ are related by*

$$\omega' \geqslant \frac{\widehat{\omega}'^2}{1 - \widehat{\omega}'} \quad \textit{and} \quad \frac{1}{2} \leqslant \widehat{\omega}' \leqslant 1.$$

Conversely, for any $(\omega', \widehat{\omega}')$ satisfying these conditions, there exists (ξ, η) with

$$\omega\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \omega' \quad \textit{and} \quad \widehat{\omega}\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \widehat{\omega}'.$$

The next open problem has been raised by M. Laurent:

Open Problem 67 (Laurent) *Is there an extension of Jarník's equality (63) in higher dimensions relating $\widehat{\omega}(\theta)$ and $\widehat{\omega}({}^t\theta)$ for $\theta \in \mathbf{R}^n$?*

3.5 Approximation by hypersurfaces

In dimension 1 an irreducible hypersurface is nothing else than a point. The exponents $\omega_n(\xi)$ and their hat companions in § 2.3 measure $|P(\xi)|$ for $P \in \mathbf{Z}[X]$, while $\omega_n^*(\xi)$ of § 2.5 measure the distance between a point $\xi \in \mathbf{C}$ and algebraic numbers α.

A generalisation of these questions in higher dimensions, where $\xi \in \mathbf{C}^n$, is the study of $|P(\xi)|$ for $P \in \mathbf{Z}[X_1, \dots, X_n]$ and of $\min_\alpha |\xi - \alpha|$ where α runs over the set of zeros of such P. As already mentioned in the introduction of § 3, a lower bound for $|P(\xi)|$ when the degree of P is fixed is nothing else than a linear independence measure for $(\xi, \dots, \xi^n)$. To consider such quantities also when the degree of P varies yields a generalisation of Mahler's classification to several variables, which has been considered by Yu Kunrui [140]. A generalisation to higher dimensions of both Mahler and Koksma classifications has been achieved by W.M. Schmidt in [126] who raises a number of open problems suggesting that the close connection between the two classifications in dimension 1 does not extend to the classification of tuples.

In [127] W.M. Schmidt deals with approximation to points ξ in $\mathbf{R}^n$ or in $\mathbf{C}^n$ by algebraic hypersurfaces, and more generally by algebraic varieties, defined over the rationals.

Let $\mathcal{M}$ be a nonempty finite set of monomials in $x_1, \dots, x_n$ with $|\mathcal{M}|$ elements. Denote by $\mathcal{P}(\mathcal{M})$ the set of polynomials in $\mathbf{Z}[x_1, \dots, x_n]$ which are linear combinations of monomials in $\mathcal{M}$. Using Dirichlet's box principle or Minkowski's theorem on linear forms, one shows the existence of nonzero elements in $\mathcal{P}(\mathcal{M})$ for which $|P(\xi)|$ is small. It is a much more difficult task to get the existence of nonzero elements in $\mathcal{P}(\mathcal{M})$ for which the distance $\delta\bigl(\xi, A(P)\bigr)$ between ξ and the hypersurface $A(P)$ defined by $P = 0$ is small.

W.M. Schmidt asks whether given ξ and $\mathcal{M}$, there exists $c = c(\xi, \mathcal{M}) > 0$ such that there are infinitely many $P \in \mathcal{P}(\mathcal{M})$ with $\delta\bigl(\xi, A(P)\bigr) \leqslant cH(P)^{-m}$, where $m = |\mathcal{M}|$ in the real case $\xi \in \mathbf{R}^n$ and $m = |\mathcal{M}|/2$ in the complex case $\xi \in \mathbf{C}^n$. He proves such an estimate when $|\mathcal{M}| = n+1$, and also in the real case when $|\mathcal{M}| = n+2$. In the case $|\mathcal{M}| = n+1$ he proves a uniform result, in the sense of Y. Bugeaud and M. Laurent [41]: given $N \geqslant 1$, there is a $P \in \mathcal{P}(\mathcal{M})$ with height $H(P) \leqslant N$ for which $\delta\bigl(\xi, A(P)\bigr) \leqslant cN^{1-m}H(P)^{-1}$. A number of further results are proved in which the exponent is not the conjectured one. The author also investigates the approximation by algebraic hypersurfaces (another reference on this topic is [94]).

Special cases of the very general and deep results of this paper were due to F. Amoroso, W.D. Brownawell, M. Laurent and D. Roy, P. Philippon. Further previous results related with Wirsing's conjecture were also achieved by V.I. Bernik and K.I. Tishchenko.

An upper bound for the distance $\delta(\xi, A)$ means that there is a point on the hypersurface A (or more generally the variety A) close to ξ. The author also investigates the "size" of the set of such elements. The auxiliary results proved in [127] on this question have independent interest.

3.6 Further metrical results

The answer to the question of Schmidt on approximation of points $\xi \in \mathbb{R}^m$ by algebraic hypersurfaces $A(P)$ is almost surely affirmative. This follows from a general theorem established by Beresnevich, Bernik, Kleinbock and Margulis. With reference to section 3.5, let $\mathcal{M}$ be a set of monomials of cardinality $m = |\mathcal{M}|$ in variables $x_1, \dots, x_k$, where we naturally assume that $m \geqslant 2$. Further, let $P(\mathcal{M})$ be the set of polynomials in $\mathbb{Z}[x_1, \dots, x_k]$ which are linear combinations of monomials in $\mathcal{M}$. Given a function $\Psi : \mathbb{N} \to (0, +\infty)$, let

$$\mathcal{A}_k(\Psi, \mathcal{M}) = \Big\{ (\xi_1, \dots, \xi_k) \in [0,1]^k : |P(\xi_1, \dots, \xi_k)| < H(P)^{-m+2} \Psi(H(P)) \\ \text{for infinitely many } P \in P(\mathcal{M}) \Big\}.$$

We are interested in $|\mathcal{A}_k(\Psi, \mathcal{M})|$, the k-dimensional Lebesgue measure of $\mathcal{A}_k(\Psi, \mathcal{M})$.

Theorem 68 (Beresnevich, Bernik, Kleinbock and Margulis) *For any decreasing Ψ,*

$$|\mathcal{A}_k(\Psi, \mathcal{M})| = \begin{cases} 0 \text{ if } \sum_{h=1}^{\infty} \Psi(h) < \infty\,, \\ 1 \text{ if } \sum_{h=1}^{\infty} \Psi(h) = \infty\,. \end{cases}$$

The convergence case of this theorem has been independently established by Beresnevich [15] and Bernik, Kleinbock and Margulis [27] using different techniques. The multiplicative analogue of the convergence part of Theorem 68, where $H(P)$ is replaced with $H^{\times}(P)$, has also been obtained in [27]. In addition, Theorem 68 holds when $\mathcal{M}$ is a set of m analytic functions defined on $(0,1)^k$ and linearly independent over $\mathbb{R}$. The analyticity assumption can also be relaxed towards a non–degeneracy condition.

The divergence case is established in [19] in the following stronger form connected with §3.5, where the notation δ and $A(P)$ are explained.

Theorem 69 (Beresnevich, Bernik, Kleinbock and Margulis) *Let Ψ be decreasing and such that $\sum_{h=1}^{\infty} \Psi(h)$ diverges. Then for almost all $\xi = (\xi_1, \dots, \xi_k) \in [0,1]^k$,*

$$\delta(\xi, A(P)) < H(P)^{-m+1} \Psi(H(P))$$

has infinitely many solutions $P \in P(\mathcal{M})$.

Taking $\Psi(h) = h^{-1} \log^{-1} h$, we get the following corollary which answers Schmidt's question in §3.5 in the affirmative for almost all points:

Corollary 1. *For almost all $\xi = (\xi_1, \dots, \xi_k) \in \mathbb{R}^k$, the inequality*

$$\delta(\xi, A(P)) < H(P)^{-m} \log^{-1} H(P) \tag{70}$$

has infinitely many solutions $P \in P(\mathcal{M})$.

Another interesting corollary corresponds to the special case of $\mathcal{M}$ being the set of all monomials of degree at most d. In this case we simply have the case of approximation by multivariable polynomials of degree at most d, where now

$$m = \binom{k+d}{d}.$$

In the case of convergence in Theorem 68, a lower bound for the Hausdorff dimension of $\mathcal{A}_k(\Psi, \mathcal{M})$ is implied by a general theorem for manifolds of Dickinson and Dodson [53]. Obtaining the corresponding upper bound in general remains an open problem but see Theorem 56 and [12, 17, 54]. The Hausdorff measure version of Theorem 69 has been established in [21].

Yet another class of interesting problems concerns the measure of transcendence and algebraic independence of numbers. Recall that complex numbers $z_1, \ldots, z_m$ are called algebraically independent if, for any non–zero polynomial $P \in \mathbb{Z}[x_1, \ldots, x_m]$, the value $P(z_1, \ldots, z_m)$ is not 0. Actually, $P(z_1, \ldots, z_m)$ can still get very small when $z_1, \ldots, z_m$ are algebraically independent. Indeed, using Dirichlet's Pigeonhole principle, one can readily show that there is a constant $c_1 > 0$ such that for any real numbers $x_1, \ldots, x_m$, there are infinitely many polynomials $P \in \mathbb{Z}[x_1, \ldots, x_m]$ such that

$$|P(x_1, \ldots, x_m)| < e^{-c_1 t(P)^{m+1}}, \tag{71}$$

where $t(P) = \deg P + \log H(P)$ is called the type of P. A conjecture of Mahler [97] proved by Nesterenko [100] says that in the case $m = 1$ for almost all real numbers x_1 there is a constant $c_0 > 0$ such that $|P(x_1)| > e^{-c_0 t(P)^2}$ for all non–zero $P \in \mathbb{Z}[x]$. Nesterenko has also shown that for $\tau = m + 2$, for almost all $(x_1, \ldots, x_m) \in \mathbb{R}^m$ there is a constant $c_0 > 0$ such that

$$|P(x_1, \ldots, x_m)| > e^{-c_0 t(P)^\tau} \qquad \text{for all non–zero } P \in \mathbb{Z}[x_1, \ldots, x_m] \tag{72}$$

and conjectured that the latter is indeed true with the exponent $\tau = m + 1$. This has been verified by Amoroso [6] over $\mathbb{C}$ but the 'real' conjecture has been recently established by Nesterenko's student Mikhailov [99]:

Theorem 73 (Mikhailov, 2007) *Let $\tau = m + 1$. Then for almost all $(x_1, \ldots, x_m) \in \mathbb{R}^m$ there is a constant $c_0 > 0$ such that (72) holds.*

We conclude by discussing the interaction of metrical, analytic and other techniques in the question of counting and distribution of rational points near a given smooth planar curve Γ. In what follows we will assume that the curvature of Γ is bounded between two positive constants. Let $N_\Gamma(Q, \delta)$ denote the number of rational points $(p_1/q, p_2/q)$, where $p_1, p_2, q \in \mathbb{Z}$ with $0 < q \leqslant Q$, within a distance at most δ from Γ.

Huxley [71] has proved that for any $\varepsilon > 0$, $N_\Gamma(Q, \delta) \ll Q^{3+\varepsilon}\delta + Q$. Until recently, this bound has remained the only non–trivial result. Furthermore, very little has been known about the existence of rational points near planar curves for $\delta < Q^{-3/2}$, that is whether $N_\Gamma(Q,\delta) > 0$ when $\delta < Q^{-3/2}$. An explicit question of this type motivated by Elkies [58] has been recently raised by Barry Mazur who asks: *"given a smooth curve in the plane, how near to it can a point with rational coordinates get and still miss?"* (Question (3) in [98, § 11]). When $\delta = o(Q^{-2})$, the rational points in question cannot miss Γ if Γ is a rational quadratic curve in the plane (see [26]). This leads to $N_\Gamma(Q,\delta)$ vanishing for some choices of Γ when $\delta = o(Q^{-2})$. For example, the curve Γ given by $x^2+y^2=3$ has no rational points [22]. When $\delta \gg Q^{-2}$, a lower bound on $N_\Gamma(Q,\delta)$ can be obtained using Khintchine's transfer principle. However, such a bound would be far from being close to the heuristic count of $Q^3\delta$ (see [26]). The first sharp lower bound on $N_\Gamma(Q,\delta)$ has been given by Beresnevich [14], who has shown that for the parabola $\Gamma = (x,x^2)$, $N_\Gamma(Q,\delta) \gg Q^3\delta$ when $\delta \gg Q^{-2}$.

Recently, Beresnevich, Dickinson and Velani [22] have shown that for arbitrary smooth planar curve Γ with non–zero curvature $N_\Gamma(Q,\delta) \gg Q^3\delta$ when $\delta \gg Q^{-2}$. Moreover, they show that the rational points in question are uniformly distributed in the sense that they form a ubiquitous system (see [18] for a discussion on ubiquity and related notions). They further apply this to get various metric results about simultaneous approximation to points on Γ. These include a Khintchine-type theorem and its Hausdorff measure analogue. In particular, for any $w \in (1/2,1)$ they explicitly obtain the Hausdorff dimension of the set of w-approximable points on Γ:

$$\dim\Big\{(x,y)\in\Gamma : \max\{\|qx\|,\|qy\|\} < q^{-w} \text{ for infinitely many } q\in\mathbb{N}\Big\} = \frac{2-w}{1+w}\cdot \qquad (74)$$

Here Γ is a smooth planar curve with non–vanishing curvature. Using analytic methods, Vaughan and Velani [137] have shown that $\varepsilon > 0$ can be removed from Huxley's estimate for $N_\Gamma(Q,\varepsilon)$. Combining the results of [22] and [137] gives the following natural generalisation of Khintchine's theorem.

Theorem 75 (Beresnevich, Dickinson, Vaughan, Velani) *Let $\psi : \mathbb{N} \to (0,+\infty)$ be monotonic. Let Γ be a $C^{(3)}$ planar curve of finite length ℓ with non–vanishing curvature and let*

$$\mathcal{A}_2(\psi,\Gamma) = \Big\{(x,y)\in\Gamma : \max\{\|qx\|,\|qy\|\} < \psi(q) \text{ holds for infinitely many } q\in\mathbb{N}\Big\}.$$

Then the arclength[3] *$|\mathcal{A}_2(\psi,\Gamma)|$ of $\mathcal{A}_2(\psi,\Gamma)$ satisfies*

$$|\mathcal{A}_2(\psi,\Gamma)| = \begin{cases} 0 \text{ if } \sum_{h=1}^\infty h\,\psi(h) < \infty\,, \\ \ell \text{ if } \sum_{h=1}^\infty h\,\psi(h) = \infty\,. \end{cases}$$

[3] one-dimensional Lebesgue measure on Γ

Furthermore, let $s \in (0,1)$ *and let* $\mathcal{H}^s$ *denote the* s*-dimensional Hausdorff measure. Then*

$$\mathcal{H}^s(\mathcal{A}_2(\psi,\Gamma)) = \begin{cases} 0 \quad \text{if } \sum_{h=1}^{\infty} h^{2-s}\psi(h)^s < \infty\,, \\ +\infty \text{ if } \sum_{h=1}^{\infty} h^{2-s}\psi(h)^s = \infty\,. \end{cases} \tag{76}$$

Note that (74) is a consequence of (76).

In higher dimensions, Druţu [55] has studied the distribution of rational points on non–degenerate rational quadrics in $\mathbb{R}^n$ and obtained a result similar to (76) in the case when $\psi(q) = o(q^{-2})$. However, simultaneous Diophantine approximation on manifolds as well as the distribution of rational points near manifolds (in particular algebraic varieties) is little understood. In other words, the higher-dimensional version of the 'near-misses' question of Mazur mentioned above has never been systematically considered.

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