ARITHMETIC DUALITY FOR p-ADIC PRO-ÉTALE COHOMOLOGY OF ANALYTIC CURVES

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Abstract. We prove a Poincaré duality for arithmetic p-adic pro-étale cohomology of smooth dagger curves over finite extensions of $\mathbf{Q}_p$. We deduce it, via the Hochschild-Serre spectral sequence, from geometric comparison theorems combined with Tate and Serre dualities. The compatibility of all the products involved is checked via reduction to the ghost circle, for which we also prove a Poincaré duality (showing that it behaves like a proper smooth analytic variety of dimension 1/2). Along the way we study functional analytic properties of arithmetic p-adic pro-étale cohomology and prove that the usual cohomology is nuclear Fréchet and the compactly supported one – of compact type.

Contents

1. Introduction

This paper is a contribution to the study of dualities in p-adic pro-étale cohomology of analytic varieties.

1.1. Statement of the main theorem.

Let p be a prime and let K be a finite extension of $\mathbf{Q}_p$. Our main theorem is the following duality result.

Theorem 1.1. (Arithmetic Poincaré duality) *Let X be a smooth, geometrically irreducible, dagger variety*[1] *of dimension* 1 *over K. Assume that X is proper, Stein, or affinoid. Then:*

(1) *There is a natural trace map isomorphism of solid $\mathbf{Q}_p$-vector spaces*[2]

$$\operatorname{Tr}_X : H^4_{\text{proét},c}(X, \mathbf{Q}_p(2)) \xrightarrow{\sim} \mathbf{Q}_p. \tag{1.2}$$

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[1]Dagger varieties, introduced by Grosse-Klönne in [22], are rigid analytic varieties with overconvergent structure sheaves.

[2]See Chapter 5 for a definition of compactly supported pro-étale cohomology.

(2) *The pairing*

$$H^i_{\text{proét}}(X,\mathbf{Q}_p(j))\otimes^{\square}_{\mathbf{Q}_p} H^{4-i}_{\text{proét},c}(X,\mathbf{Q}_p(2-j))\to H^4_{\text{proét},c}(X,\mathbf{Q}_p(2))\xrightarrow[\sim]{\mathrm{Tr}_X}\mathbf{Q}_p$$

is a perfect duality, i.e., we have induced isomorphisms of solid $\mathbf{Q}_p$-vector spaces

$$\gamma_{X,i}: H^i_{\text{proét}}(X,\mathbf{Q}_p(j))\xrightarrow{\sim} H^{4-i}_{\text{proét},c}(X,\mathbf{Q}_p(2-j))^*,$$
$$\gamma^c_{X,i}: H^i_{\text{proét},c}(X,\mathbf{Q}_p(j))\xrightarrow{\sim} H^{4-i}_{\text{proét}}(X,\mathbf{Q}_p(2-j))^*,$$

where $(-)^*$ denotes $\underline{\mathrm{Hom}}_{\mathbf{Q}_p}(-,\mathbf{Q}_p)$.

Remark 1.3. (i) If the curve X is proper then the cohomology groups $H^i_{\text{proét}}(X,\mathbf{Q}_p(j))$ are of finite rank over $\mathbf{Q}_p$; this is also the case if $j\neq 1$ and if X is affinoid (or Stein with $H^1_{\mathrm{dR}}(X)$ finite dimensional).

(ii) If X is Stein, $H^i_{\text{proét}}(X,\mathbf{Q}_p(j))$ is a nuclear[3] Fréchet and $H^i_{\text{proét},c}(X,\mathbf{Q}_p(j))$ is a space of compact type; if X is a dagger affinoid – it is the other way around.

Remark 1.4. If X is proper or Stein we have a more general derived duality: the duality map

$$\gamma_X: \mathrm{R}\Gamma_{\text{proét}}(X,\mathbf{Q}_p(j))\to \mathbb{D}(\mathrm{R}\Gamma_{\text{proét},c}(X,\mathbf{Q}_p(2-j))[4]), \tag{1.5}$$

where $\mathbb{D}(-)=\mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{Q}_p}(-,\mathbf{Q}_p)$, is a quasi-isomorphism in $\mathscr{D}(\mathbf{Q}_{p,\square})$. The isomorphisms $\gamma_{X,i}$ are obtained from it because all the higher Ext groups involved vanish since $H^i_{\text{proét},c}(X,\mathbf{Q}_p(j))$ is of compact type. The isomorphisms $\gamma^c_{X,i}$ are obtained from $\gamma_{X,i}$ by dualizing using the fact that the spaces $H^i_{\text{proét},c}(X,\mathbf{Q}_p(j))$ are (solid) reflexive.

We venture to conjecture an arithmetic Poincaré duality in any dimension:

Conjecture 1.6. *Let X be a smooth Stein dagger variety over K, geometrically irreducible, of dimension d. Then:*

(1) *The cohomology groups $H^i_{\text{proét}}(X,\mathbf{Q}_p(j))$ and $H^i_{\text{proét},c}(X,\mathbf{Q}_p(j))$ are nuclear Fréchet and of compact type, respectively.*
(2) *We have (quasi-)isomorphisms in $\mathscr{D}(\mathbf{Q}_{p,\square})$*

$$\mathrm{R}\Gamma_{\text{proét}}(X,\mathbf{Q}_p(j))\simeq \mathbb{D}(\mathrm{R}\Gamma_{\text{proét},c}(X,\mathbf{Q}_p(d+1-j))[2d+2]),$$
$$H^i_{\text{proét}}(X,\mathbf{Q}_p(j))\simeq H^{2d+2-i}_{\text{proét},c}(X,\mathbf{Q}_p(d+1-j))^*,$$
$$H^i_{\text{proét},c}(X,\mathbf{Q}_p(j))\simeq H^{2d+2-i}_{\text{proét}}(X,\mathbf{Q}_p(d+1-j))^*.$$

Example 1.7. The starting point of our study of dualities for p-adic pro-étale cohomology of analytic spaces was the computation for $X=D$, the open unit disc. For example, we computed that:

$$H^1_{\text{proét}}(X,\mathbf{Q}_p(1))\simeq (\mathscr{O}(D)/K)\oplus H^1(\mathscr{G}_K,\mathbf{Q}_p(1)),$$
$$H^3_{\text{proét},c}(X,\mathbf{Q}_p(1))\simeq (\mathscr{O}(\partial D)/\mathscr{O}(D))\oplus H^1(\mathscr{G}_K,\mathbf{Q}_p),$$

where $\mathscr{G}_K=\mathrm{Gal}(\overline{K}/K)$ and $\partial D:=\varprojlim_n D\setminus D_n$ is the "ghost circle", boundary of the unit disk (here, D_n is the closed ball $\{v_p(z)\geq \frac{1}{n}\}$). The splitting is not canonical but it is compatible with products. This computation showed that there is a duality of the form stated in Theorem 1.1 and, moreover, that it is induced by Galois duality and coherent duality (i.e., Serre duality):

$$H^i(\mathscr{G}_K,\mathbf{Q}_p)\simeq H^{2-i}(\mathscr{G}_K,\mathbf{Q}_p(1))^*,$$
$$H^0(D,\Omega^1_D)\simeq H^1_c(D,\mathscr{O}_D)^*.$$

[3]Nuclear in the classical sense not in the solid sense.

We used here the isomorphisms (the first one is almost the definition of $H^1_c(D,\mathscr{O}_D)$, taking into account vanishing of $H^2(D,\mathscr{O}_D)$; the second is induced by $f\mapsto df$: that it is an isomorphism is equivalent to the fact that $H^1_{\mathrm{dR}}(D)=0$):

$$H^1_c(D,\mathscr{O}_D)\xleftarrow{\sim}\mathscr{O}(\partial D)/\mathscr{O}(D),\quad H^0(D,\Omega^1_D)\xleftarrow{\sim}\mathscr{O}(D)/K. \tag{1.8}$$

Since $[K:\mathbf{Q}_p]<\infty$, the above coherent K-duality can be turned into a $\mathbf{Q}_p$-duality by composing with $\mathrm{Tr}_{K/\mathbf{Q}_p}$.

Remark 1.9. The above results for the open disk do not involve the solid formalism, and we attempted at first to write this paper using classical functional analysis as in [15]. We passed to solid formalism because of two reasons. One was a need for a derived dual that would work with Mayer-Vietoris sequences and which does not seem to exist in the classical world in the generality we wanted. The second one was a construction of topological Hochschild-Serre spectral sequences which we did not know how to do in the classical set-up.

1.2. **The proof of Theorem 1.1.** An analog of Theorem 1.1 holds for schemes and the proof uses Galois descent from geometric, i.e., over $C:=\widehat{\overline{K}}$, Poincaré duality and Galois duality (the Hochschild-Serre spectral sequence does not degenerate at E_2 but it does degenerate at E_3). An analogous argument, starting with geometric Poincaré duality due to Zavyalov [34], Gabber, and Mann [25], yields Theorem 1.1 for proper rigid analytic varieties (in fact, in any dimension).

We use here a similar strategy. In our setting we do not yet have the geometric duality so we replace it with comparison isomorphisms. They allow us to pass from geometric p-adic pro-étale cohomology to de Rham data and the terms of the Hochschild-Serre spectral sequence can be identified with coherent cohomology and Galois cohomology of $\mathbf{Q}_p$-vector spaces with some finiteness properties. Our Poincaré duality is now deduced, via this spectral sequence, from coherent and Galois dualities.

• *Geometric comparison results.* Recall that, for smooth Stein rigid analytic varieties over K, we have comparison theorems between geometric p-adic pro-étale cohomology and geometric syntomic cohomology. In the case of usual cohomology they are due to Colmez-Dospinescu-Nizioł [15] and Colmez-Nizioł [18]; in the case of compactly supported cohomology – to Achinger-Gilles-Nizioł [1].

In the case where X is a smooth, geometrically irreducible, Stein curve over K, they yield:

(1) Vanishings:

$$\begin{aligned} H^i_{\mathrm{pro\acute{e}t}}(X_C,\mathbf{Q}_p)&=0,\quad \text{for } i\neq 0,1,\\ H^i_{\mathrm{pro\acute{e}t},c}(X_C,\mathbf{Q}_p)&=0,\quad \text{for } i\neq 1,2.\end{aligned} \tag{1.10}$$

(2) Isomorphisms:

$$H^0_{\mathrm{pro\acute{e}t}}(X_C,\mathbf{Q}_p)\simeq\mathbf{Q}_p,\quad H^1_{\mathrm{pro\acute{e}t},c}(X_C,\mathbf{Q}_p(1))\xrightarrow{\sim}\mathrm{HK}^1_c(X_C,1),$$

where[4] $\mathrm{HK}^j_*(X_C,i):=(H^j_{\mathrm{HK},*}(X_C)\otimes^{\square}_{\breve{F}}\widehat{\mathbf{B}}^+_{\mathrm{st}})^{N=0,\varphi=p^i}$ and $H^i_{\mathrm{HK},*}(X_C)$, for $*=[\ ],c$, is the Hyodo-Kato cohomology.

(3) Exact sequences:

$$\begin{aligned} 0\to\mathscr{O}(X_C)/C\to &H^1_{\mathrm{pro\acute{e}t}}(X_C,\mathbf{Q}_p(1))\to\mathrm{HK}^1(X_C,1)\to 0\\ \mathrm{HK}^1_c(X_C,2)\to H^1\mathrm{DR}_c(X_C,2)\to &H^2_{\mathrm{pro\acute{e}t},c}(X_C,\mathbf{Q}_p(2))\to\mathbf{Q}_p(1)\to 0,\end{aligned} \tag{1.11}$$

where we set

$$\mathrm{DR}_c(X_C,i):=(H^1_c(X,\mathscr{O}_X)\otimes^{\square}_K(\mathbf{B}^+_{\mathrm{dR}}/F^i)\to H^1_c(X,\Omega^1_X)\otimes^{\square}_K(\mathbf{B}^+_{\mathrm{dR}}/F^{i-1}))[-1],$$

[4]Here $\breve{F}$ is the completion of the maximal unramified extension of the fraction field F of the Witt vectors of the residue field of K and $\widehat{\mathbf{B}}^+_{\mathrm{st}}$ is the semistable period ring of Fontaine in its Banach form.

The last sequence yields the definition of the geometric trace map

$$\mathrm{Tr}_{X_C} : H^2_{\mathrm{pro\acute{e}t},c}(X_C, \mathbf{Q}_p(1)) \to \mathbf{Q}_p.$$

The arithmetic trace map (1.2) is defined as the composition

$$\mathrm{Tr}_X : H^4_{\mathrm{pro\acute{e}t},c}(X, \mathbf{Q}_p(2)) \simeq H^2(\mathscr{G}_K, H^2_{\mathrm{pro\acute{e}t},c}(X, \mathbf{Q}_p(2))) \xrightarrow{\mathrm{Tr}_{X_C}(1)} H^2(\mathscr{G}_K, \mathbf{Q}_p(1)) \xrightarrow{\mathrm{Tr}_K} \mathbf{Q}_p.$$

• *Galois descent.* For $s \in \mathbf{Z}$, we have Hochschild-Serre spectral sequences

$$E_2^{i,j} = H^i(\mathscr{G}_K, H^j_{\mathrm{pro\acute{e}t},*}(X_C, \mathbf{Q}_p(s))) \Rightarrow H^{i+j}_{\mathrm{pro\acute{e}t},*}(X, \mathbf{Q}_p(s)). \tag{1.12}$$

The computations done above show that to understand the terms of the spectral sequences (1.12) we need to understand Galois cohomology of

$$H^0(X, \Omega^1_X) \otimes^{\square}_K C(s), \quad H^1_c(X, \mathscr{O}_X) \otimes^{\square}_K C(s), \quad \mathrm{HK}^1_c(X_C, 1)(s). \tag{1.13}$$

We used here the isomorphisms (1.8). We claim that, in the case $H^1_{\mathrm{HK}}(X_C)$ is of finite rank over $\breve{F}$, this cohomology is or trivial or isomorphic to

$$H^0(X, \Omega^1_X), \quad H^1_c(X, \mathscr{O}_X), \quad H^i(\mathscr{G}_K, V), \tag{1.14}$$

where V is a finite rank $\mathbf{Q}_p$-Galois representation. Indeed, note that the last group in (1.13) is an almost C-representation [20] (a finite rank C-vector space plus/minus a finite rank $\mathbf{Q}_p$-vector space). Our claim then follows easily from a generalization of Tate's computations:

$$H^i(\mathscr{G}_K, C(j)) \simeq \begin{cases} K & \text{if } j = 0, i = 0, 1; \\ 0 & \text{otherwise.} \end{cases}$$

By the vanishing results (1.10), the spectral sequences (1.12) have only two nontrivial columns, hence degenerate at E_2. Looking at the terms of the spectral sequences in (1.14), we see that if we knew that the arithmetic pro-étale product is compatible with the coherent and Galois products we would have our duality (at least in the case the Hyodo-Kato cohomology is of finite rank but we can always reduce to that case). However, we have found it very difficult to check this compatibility for a general curve X as above: the main difficulty arises from the exact sequence (1.11) that does not behave well with respect to products. We decided thus to pass from X to simpler curves.

Nevertheless, the above computations imply the following functional analytic result:

Theorem 1.15. *Let X be a smooth Stein dagger curve over K, geometrically irreducible. Then the cohomology groups $H^i_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(j))$ and $H^i_{\mathrm{pro\acute{e}t},c}(X, \mathbf{Q}_p(j))$ are nuclear Fréchet and of compact type, respectively.*

This is because we know these properties for $H^0(X, \Omega^1_X)$ and $H^1_c(X, \mathscr{O}_X)$, respectively, hence for the terms of the Hochschild-Serre spectral sequences, and the extension problems arising from the spectral sequences can be solved.

• *Reductions.* By a simple limit argument, to prove the derived Poincaré duality (1.5), we can pass from a general Stein curve to a wide open curve, i.e., to a complement of a finite number of closed discs in a proper curve (we allow for a finite field extension here). For wide open curves we use a Mayer-Vietoris argument to reduce to proper curves, open discs, and open annuli. For proper curves we know the result, for open discs and annuli we reduce the computation to the one of their boundaries.

• *Arithmetic duality for ghost circle.* We show that, in duality theory, the boundary of an open disc, a ghost circle, behaves like a proper rigid analytic variety over K of dimension 1/2.

Theorem 1.16. (Arithmetic duality for ghost circle) *Let D be an open disc over K. Let $Y := \partial D$ be the boundary of D. Then:*

(1) There is a natural trace map isomorphism of solid $\mathbf{Q}_p$-vector spaces

$$\mathrm{Tr}_Y : H^3_{\mathrm{pro\acute{e}t}}(Y, \mathbf{Q}_p(2)) \xrightarrow{\sim} \mathbf{Q}_p.$$

(2) *The pairing*

$$H^i_{\text{proét}}(Y, \mathbf{Q}_p(j)) \otimes^{\square}_{\mathbf{Q}_p} H^{3-i}_{\text{proét}}(Y, \mathbf{Q}_p(2-j)) \to H^3_{\text{proét}}(Y, \mathbf{Q}_p(2)) \xrightarrow{\sim} \mathbf{Q}_p \tag{1.17}$$

is a perfect duality, i.e., we have the induced isomorphism of solid $\mathbf{Q}_p$-vector spaces

$$\gamma_Y : \quad H^i_{\text{proét}}(Y, \mathbf{Q}_p(j)) \xrightarrow{\sim} H^{3-i}_{\text{proét}}(Y, \mathbf{Q}_p(2-j))^*.$$

We will now sketch the proof of this theorem. We define an ascending filtration on $H^i_{\text{proét}}(Y, \mathbf{Q}_p(j))$:

$$F^2_{i,j} = H^i_{\text{proét}}(Y, \mathbf{Q}_p(j)) \supset F^1_{i,j} \supset F^0_{i,j} \supset F^{-1}_{i,j} = 0,$$

such that

$$\begin{aligned} &F^2_{i,j}/F^1_{i,j} \simeq H^{i-1}(\mathscr{G}_K, \mathbf{Q}_p(j-1)),\\ &F^1_{i,j}/F^0_{i,j} \simeq H^{i-1}(\mathscr{G}_K, \mathscr{O}(Y_C)/C(j-1)),\\ &F^0_{i,j} \simeq H^i(\mathscr{G}_K, \mathbf{Q}_p(j)). \end{aligned}$$

It is helpful to visualize this filtration in the following way (with $H^i(-\mathbf{Q}_p(j)) := H^i_{\text{proét}}(-\mathbf{Q}_p(j))$):

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & F^0_{i,j} := H^i(\mathscr{G}_K, \mathbf{Q}_p(j)) \arrow[r] \arrow[d, equal] & F^1_{i,j} \arrow[r] \arrow[d] & H^{i-1}(\mathscr{G}_K, \frac{\mathscr{O}(Y_C)}{C}(j-1)) \arrow[r] \arrow[d] & 0\\
0 \arrow[r] & H^i(\mathscr{G}_K, \mathbf{Q}_p(j)) \arrow[r] & F^2_{i,j} := H^i(Y, \mathbf{Q}_p(j)) \arrow[r] \arrow[d] & H^{i-1}(\mathscr{G}_K, H^1(Y_C, \mathbf{Q}_p(j))) \arrow[r] \arrow[d] & 0\\
 & & H^{i-1}(\mathscr{G}_K, \mathbf{Q}_p(j-1)) \arrow[r, equal] \arrow[d] & H^{i-1}(\mathscr{G}_K, \mathbf{Q}_p(j-1)) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

The middle exact row comes from the filtration induced by the Hochschild-Serre spectral sequence; the right exact column is induced by the syntomic filtration from the analog[5] of the exact sequence (1.11). The term $F^1_{i,j}$ is defined as a pullback of the top right square.

Now, the key computation is the following:

Theorem 1.18. (Explicit Reciprocity Law) *We have:*

(1) *The pairing (1.17) is compatible with the above filtration.*
(2) *On the associated grading the pairing (1.17) yields a pairing induced by the Galois cohomology pairing and coherent pairing.*

The proof of this Theorem interpolates between syntomic and (φ, Γ)-techniques.

This finishes the proof of Theorem 1.16 and hence of Theorem 1.1, the main theorem of this paper.

1.3. Conjectural geometric duality.

We finish the introduction with a discussion of what we think happens over C. Let us start with the example that guided our study.

Example 1.19. Let D be the open unit disc over C. The nontrivial cohomology groups are:

$$H^0_{\text{proét}}(D, \mathbf{Q}_p(j)) \simeq \mathbf{Q}_p(j), \quad H^1_{\text{proét}}(D, \mathbf{Q}_p(j)) \simeq (\mathscr{O}(D)/C)(j-1),$$
$$H^2_{\text{proét},c}(D, \mathbf{Q}_p(j)) \simeq \mathbf{Q}_p(j-1) \oplus (\mathscr{O}(\partial D)/\mathscr{O}(D))(j-1).$$

It looks like we have the right groups for a duality (for appropriate choices of j): $\mathscr{O}(D)/C$ and $\mathscr{O}(\partial D)/\mathscr{O}(D)$ are C-vector spaces in duality via coherent duality and $\mathbf{Q}_p(j)$ and $\mathbf{Q}_p(j-1)$ can be

[5] Take this sequence for an annuli and go to the limit towards the boundary.

put in duality. But the degrees are wrong for a Poincaré-type duality to work: coherent duality adds up to degree 3 but $\mathbf{Q}_p$-vector space duality adds up to degree 2. Also, since $[C:\mathbf{Q}_p]=\infty$, it is impossible to turn a C-duality into a $\mathbf{Q}_p$-duality.

To find a set-up in which a duality could be restored, we turned to the category of Vector Spaces, i.e., v-sheaves of topological $\mathbf{Q}_p$-vector spaces on the category Perf_C of perfectoid spaces over C (that it is not unreasonnable to do so is due to the fact that there is a natural way [17] to turn pro-étale geometric cohomology groups into VS's). There we have the following computation [11, Prop. 10.16] of Ext-groups:

$$\mathrm{Hom}_{\mathrm{VS}}(\mathbf{Q}_p,\mathbf{Q}_p(1))\simeq\mathbf{Q}_p(1),\quad \mathrm{Ext}^1_{\mathrm{VS}}(\mathbf{Q}_p,\mathbf{Q}_p(1))=0,$$
$$\mathrm{Hom}_{\mathrm{VS}}(\mathbb{G}_a,\mathbf{Q}_p(1))=0,\quad \mathrm{Ext}^1_{\mathrm{VS}}(\mathbb{G}_a,\mathbf{Q}_p(1))\simeq C.$$

Moreover, Ext^i, for $i\geq 2$, vanish by [2, Th. 3.8]. The nontrivial Ext-group in the second row is generated by the fundamental exact sequence of Banach-Colmez spaces

$$0\to\mathbf{Q}_p(1)\to\mathbb{B}_{\mathrm{cr}}^{+,\varphi=p}\to\mathbb{G}_a\to 0$$

The above computations, ignoring functional analytic questions, yield a Verdier duality isomorphisms (with $H^i_*(-\mathbf{Q}_p(j)):=H^i_{\text{proét},*}(-\mathbf{Q}_p(j))$):

$$\mathrm{Ext}^1_{\mathrm{VS}}(H^2_c(D,\mathbf{Q}_p(2-j)),\mathbf{Q}_p(1))\stackrel{\sim}{\to}H^1(D,\mathbf{Q}_p(j)),$$
$$0\to\mathrm{Ext}^1_{\mathrm{VS}}(H^1(D,\mathbf{Q}_p(2-j)),\mathbf{Q}_p(1))\to H^2_c(D,\mathbf{Q}_p(j))\to\mathrm{Hom}_{\mathrm{VS}}(H^0(D,\mathbf{Q}_p(2-j)),\mathbf{Q}_p(1))\to 0$$

Guided by this, we venture a conjectural statement of geometric duality for p-adic pro-étale cohomology in any dimension (again, ignoring topology):

Conjecture 1.20. (Geometric Verdier duality) *Let X be a smooth Stein rigid analytic variety over C, connected, of dimension d. There is a natural quasi-isomorphism*

$$\mathrm{R}\Gamma_{\text{proét}}(X,\mathbf{Q}_p(j))\simeq\mathrm{RHom}_{\mathrm{VS}}(\mathrm{R}\Gamma_{\text{proét},c}(X,\mathbf{Q}_p(d+1-j))[2d],\mathbf{Q}_p(1)).$$

The interested reader can find more details in [14].

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Notation and conventions. Let K be a finite extension of $\mathbf{Q}_p$, with ring of integers $\mathscr{O}_K$ and residue field k. Let $\overline{K}$ be an algebraic closure of K and let $\mathscr{O}_{\overline{K}}$ denote the integral closure of $\mathscr{O}_K$ in $\overline{K}$. Set $\mathscr{G}_K=\mathrm{Gal}(\overline{K}/K)$. Let $C=\widehat{\overline{K}}$ be the p-adic completion of $\overline{K}$. Let $W(k)$ be the ring of Witt vectors of k with fraction field F (i.e., $W(k)=\mathscr{O}_F$); let $e=e_K$ be the ramification index of K over F. Let $\breve{F}:=W(\overline{k})[\frac{1}{p}]$ denote the completion of the maximal unramified extension of F and let φ be the absolute Frobenius on $W(\overline{k})$.

We will denote by $\mathbf{A}_{\mathrm{inf}}, \mathbf{A}_{\mathrm{cr}}, \mathbf{B}_{\mathrm{cr}}, \widehat{\mathbf{B}}_{\mathrm{st}}, \mathbf{B}_{\mathrm{dR}}$ the Witt, crystalline, semistable, and de Rham period rings of Fontaine, respectively.

All rigid analytic spaces and dagger spaces considered will be over K or C. We assume that they are separated, taut, and countable at infinity.

Since the only cohomology with $\mathbf{Q}_p(r)$-coefficients that we consider is the pro-étale cohomology, we will remove "proét" from the notations, i.e. write:

$$H^i(-,\mathbf{Q}_p(r)) := H^i_{\text{proét}}(-,\mathbf{Q}_p(r)), \quad H^i_c(-,\mathbf{Q}_p(r)) := H^i_{\text{proét},c}(-,\mathbf{Q}_p(r)).$$

2. Functional analysis

2.1. **Classical functional analysis.** We gather here some basic facts from classical p-adic functional analysis that we use in the paper.

2.1.1. *Locally convex vector spaces.* Our cohomology groups will be equipped with a canonical topology. To talk about it in a systematic way, we will work in the category C_K of locally convex K-vector spaces. For details the reader may consult [15, Sec. 2.1, 2.3]. To summarize quickly: C_K is a quasi-abelian category. We will denote the category of left-bounded complexes of C_K by $C(C_K)$ and the associated derived ∞-category by $\mathscr{D}(C_K)$. A morphism of complexes that is a quasi-isomorphism in $\mathscr{D}(C_K)$, i.e., its mapping cofiber is strictly exact, will be called a *strict quasi-isomorphism.* The associated cohomology objects are denoted by[6] $\widetilde{H}^n(E) \in LH(C_K)$:

$$\widetilde{H}^n(E) := \tau_{\leq n}\tau_{\geq n}(E) = (\mathrm{coim}(d_{n-1}) \to \ker(d_n)), \quad E \in C(C_K); \tag{2.1}$$

they are called *classical* if the canonical map $\widetilde{H}^n(E) \to H^n(E)$ is an isomorphism[7]. Classical objects are closed under extensions in $\mathrm{LH}(C_K)$.

2.1.2. *Hausdorff locally convex vector space.* Our main reference here is [28, Sec. 3.1]. We will denote by C_K^H the category of Hausdorff locally convex K-vector spaces. It is stable under direct sums and direct products and if $V \subset W$ are two locally convex K-vector spaces then W/V is Hausdorff if and only if V is closed in W. The category C_K^H is quasi-abelian: kernels and coimages are defined as in C_K; cokernels and images are defined using closures of images in C_K. A sequence in C_K^H is strictly exact if and only if it is strictly exact in C_K.

2.1.3. *Duality.* If $V, W \in C_K$, we denote by $\mathscr{L}(V,W)$ the space of continuous linear maps from V to W. We write $\mathscr{L}_s(V,W)$ and $\mathscr{L}_b(V,W)$ for the vector space $\mathscr{L}(V,W)$ equipped with the weak and strong topologies (i.e., the topology of pointwise convergence and the topology of bounded convergence), respectively. For $V \in C_K$, we set $V'_s := \mathscr{L}_s(V,K)$ and $V^* := V'_b := \mathscr{L}_b(V,K)$. Both of these dual spaces are Hausdorff. If V is a Hausdorff space it is called *reflexive* if the duality map $V \to (V'_b)'_b$ is a topological isomorphism.

We will also use *stereotype dual* $V^\star$ of $V \in C_K$. It is defined as $\mathscr{L}(V,K)$ equipped with the topology of compactoid convergence[8] of V. We have continuous maps

$$V'_b \to V^\star \to V'_s.$$

If V is a Banach space then they are not topological isomorphisms unless V has finite dimension. For any Banach space, its stereotype dual is a Smith space[9], and vice versa, for any Smith space, its stereotype dual is a Banach space, and $(V^\star)^\star = V$ if V is Banach or Smith. If we write a Banach space $V \in C_K$ as $V \simeq (\widehat{\oplus_I \mathscr{O}_K})[1/p]$ then its stereotype dual is the Smith space $V^\star \simeq (\prod_I \mathscr{O}_K)[1/p]$. A Banach space is a Smith space if and only if it is finite dimensional.

[6] LH stands for "left heart".

[7] In our situations this is usually equivalent to $H^n(E)$ being separated.

[8] See Section 2.1.5 for a definition of compactoids.

[9] A *Smith space* is a complete compactly generated locally convex vector space V having a universal compact set, i.e., a compact set K, which absorbs every other compact set $T \subseteq V$ (i.e., $T \subseteq \lambda \cdot K$, for some $\lambda > 0$).

2.1.4. *Hausdorff compactly generated locally convex vector spaces.* Recall that a topological space T is *compactly generated* if a map $f : T \to T'$ to another topological space is continuous as soon as the composite $S \to T \to T'$ is continuous for all compact Hausdorff spaces S mapping to T. The inclusion of compactly generated spaces into all topological spaces admits a right adjoint $T \mapsto T^{\rm cg}$ that sends a topological space T to its underlying set equipped with the quotient topology for the map $\coprod_{S\to T} S \to T$, where the disjoint union runs over all compact Hausdorff spaces S (alternatively, profinite sets S) mapping to T.

Any first-countable space (in particular, any metrizable topological space) is compactly generated (see [8, Remark 1.6] for a proof). So, for example, Fréchet spaces are compactly generated. The category of compactly generated spaces is closed under taking coproducts, closed subspaces and quotients by closed subspaces. Hence a colimit of Fréchet spaces is compactly generated if it is Hausdorff; this applies, in particular, to locally convex vector spaces of compact type[10] since they are Hausdorff and can be written as a countable colimit of Banach spaces.

The category $C_K^{\rm Hcg}$ of Hausdorff compactly generated locally convex vector spaces over K is quasi-abelian: kernels and coimages are defined as in C_K; cokernels and images are defined using closures of images in C_K. A sequence in $C_K^{\rm Hcg}$ is strictly exact if and only if it is strictly exact in C_K.

2.1.5. *Spaces of compact type and nuclear spaces.* Our references for this section are [30, IV.19], [31, Ch. 1], [27, Ch. 8].

Definition 2.2. Let $V, W \in C_K$ be Hausdorff.

(1) A subset $B \subset V$ is called *compactoid* if for any open lattice $L \subset V$ there are finitely many vectors $v_1, \ldots, v_m \in V$ such that $B \subset L + \mathscr{O}_K v_1 + \cdots + \mathscr{O}_K v_m$. Compactoids are preserved by maps in C_K.
(2) A continuous linear map $f : V \to W$ is called *compact* if there is an open lattice L in V such that the closure of $f(L)$ in W is compactoid and complete. If W is quasi-complete (in particular, if it is complete) then this is equivalent to $f(L)$ being compactoid.
(3) V is called of *compact type* if it is the inductive limit of a sequence

$$V_1 \xrightarrow{\iota_1} V_2 \xrightarrow{\iota_2} V_3 \to \cdots$$

of Hausdorff spaces $V_n \in C_K$, for $n \in \mathbf{N}$, with injective compact linear maps. By the proof of [30, Prop. 16.10], we may assume the V_n's to be Banach spaces.

We will use often the following facts (see [30, Rem. 16.7]).

Lemma 2.3. *Let $g : V \to W$ be a compact map. Then*

(1) If $h : V_1 \to V$ and $f : W \to W_1$ are arbitrary continuous linear maps then the map $fgh : V_1 \to W_1$ is compact.
(2) If the image of g is contained in a closed subpace $W_0 \subset W$ then the induced map $g : V \to W_0$ is compact.

We will denote by $C_{c,K}$ the full subcategory of C_K consisting of spaces of compact type. Spaces of compact type are Hausdorff, complete, and reflexive. Their strong duals are Fréchet and satisfy $V'_b = \varprojlim_n (V'_n)_b$. Moreover, a closed subspace of a space of compact type is also of compact type and so is the relevant quotient; $C_{c,K}$ is closed under countable direct sums.

Definition 2.4. The space $V \in C_K$ is called *nuclear* if for any open lattice $L \subset V$ there exists another open lattice $M \subset L$ such that the canonical map between completions $\widehat{V}_M \to \widehat{V}_L$ is compact (i.e., the image of $M \to L/p^L$ is, for any n, contained in a module of finite type over $\mathscr{O}_K$).

[10] See Section 2.1.5 below for the definition of spaces of compact type.

Nuclear Banach spaces are finite dimensional. Nuclear Fréchet spaces are reflexive. Moreover, a subspace of a nuclear space is nuclear and if this subspace is closed the relevant quotient is nuclear. A countable inductive limit of nuclear spaces is nuclear (see [27, Th. 8.5.7]). By loc. cit., projective limits of nuclear spaces are nuclear. Any compact projective or inductive limit[11] of locally convex K-vector spaces is nuclear. A Fréchet space is the strong dual of a space of compact type if and only if it is nuclear.

The following lemma will be essential for us.

Lemma 2.5. *(1) Every strict exact sequence of spaces from C_K*

$$0 \to V \to W \to W' \to 0$$

such that V is of finite rank and W' is Hausdorff splits (and $W \simeq V \oplus W'$).

(2) In a strict extension of spaces from C_K

$$0 \longrightarrow V \longrightarrow W \longrightarrow W' \longrightarrow 0, \tag{2.6}$$

where V is of finite rank over K and W' is nuclear Fréchet, W is also nuclear Fréchet. Same holds for spaces of compact type.

Proof. In the first claim, we may assume that V is of rank 1. Note that W is Hausdorff. Take a continuous linear form $\lambda : W \to K$, not identically 0 on V (such a form exists by [30, Cor. 9.3]). It suffices to show that the canonical map $V \oplus \operatorname{Ker}\lambda \to W$ is a topological isomorphism.

It is an algebraic isomorphism and it is continuous. Hence we just need to check that it is open. So, let U_λ be an open lattice in $\operatorname{Ker}\lambda$ and let V_0 be an open lattice in V. We need to show that $U_\lambda + V_0$ is open in W and it is enough to do it for open sub-lattices of U_λ and V_0.

Any neighborhood of 0 in $\operatorname{Ker}\lambda$ contains an open of the form $U \cap (\operatorname{Ker}\lambda)$, where U is an open lattice in W. Hence we can assume U_λ to be of this form. By construction, we have an exact sequence $0 \to U_\lambda \to U \to \lambda(U) \to 0$.

Now, U contains $p^N V_0$ for N big enough, since it is a lattice. Then $U_\lambda + p^N V_0$ is the inverse image of $p^N\lambda(V_0)$ in U by λ, hence it is open as λ is continuous and $p^N\lambda(V_0)$ is open in K. We proved what we wanted, up to replacing V_0 by its sub-lattice $p^N V_0$.

The second claim follows immediately from the first one. $\square$

Lemma 2.7. *If we have a map of strict exact sequences of complete Hausdorff spaces from C_K*

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r] \arrow[d, "f_V"] & W \arrow[r] \arrow[d, "f_W"] & W' \arrow[r] \arrow[d, "f_{W'}"] & 0 \\
0 \arrow[r] & V_1 \arrow[r] & W_1 \arrow[r] & W_1' \arrow[r] & 0
\end{tikzcd}$$

such that V, V_1 are of finite rank over K, the bottom sequence splits, and the map $f_{W'}$ is compact then the map f_W is compact as well.

Proof. Take an open lattice $L_{W'}$ in W' such that $f_{W'}(L_{W'})$ is compactoid. Let L_W be the preimage of $L_{W'}$ in W. Choose a section $s : W_1' \to W_1$ of the projection $\pi : W_1 \to W_1'$ and consider the continuous map

$$g : W \to V_1, \quad g(w) := f_W(w) - s\pi(f_W(w)).$$

Take a compact lattice L_{V_1} in V_1, its preimage (via g) in W, and then change L_W to its intersection with that preimage.

Now, we see that $f_W(L_W)$ is compactoid: we have

$$f_W(L_W) \subset g(L_W) + sf_{W'}(L_{W'})$$

and both $g(L_W)$ and $sf_{W'}(L_{W'})$ are compactoid in W_1. This concludes the proof of our lemma. $\square$

[11]In the sense of [30, Cor. 16.6, Prop. 16.10] .

Let $C_{nF,K}$ denote the full subcategory of C_K consisting of nuclear Fréchet spaces. The functor

$$C_{c,K} \to C_{nF,K}, \quad V \mapsto V'_b,$$

is an anti-equivalence of categories. For any two $V, W \in C_{c,K}$ the natural linear map $\mathscr{L}_b(V,W) \to \mathscr{L}_b(W'_b, V'_b)$ is a topological isomorphism.

Lemma 2.8. *If $X \in C_K$ is nuclear Fréchet or of compact type then the canonical map $X'_b \to X^\star$ is a topological isomorphism.*

Proof. In both cases the space X is reflexive, hence Montel (see [27, Cor. 8.4.22]). But in Montel spaces, by [27, Th. 8.4.5], every bounded subset is compactoid (and vice versa, of course) thus the strong topology on the algebraic dual of X coincides with the topology of compactoid convergence, as wanted. □

2.2. **Solid functional analysis.** We will review here briefly results from solid functional analysis that we will need. Our main references are [5], [29], [8], [9].

2.2.1. *Basic properties of condensed sets.* Let Cond denote the category of *condensed sets*, i.e., sheaves of sets on the site of pro-finite sets with coverings given by finite families of jointly surjective maps[12] or, equivalently, on the pro-étale site $*_{\text{proét}}$ of a geometric point. We define similarly condensed groups, rings, etc.

We will denote by CondAb the category of condensed abelian groups. It is an abelian category [8, Thm. 2.2]. It has all limits and colimits. Arbitrary products, arbitrary direct sums and filtered colimits are exact. It is generated by compact projective objects. For a condensed commutative ring A, we will write $\mathrm{Mod}_A^{\mathrm{cond}}$ for the category of A-modules in CondAb and $\underline{\mathrm{Hom}}_A(-,-)$ for its internal Hom (in the case $A = \mathbf{Z}$, we will often omit the subscript $\mathbf{Z}$).

Remark 2.9. Because of set theoretical issues this definition of the category Cond is not sensu stricto correct. The correct definition of Cond is given in [8, Lecture II]. In this paper we will use the latter though we find it helpful to keep in mind its simplified version given above.

For a condensed set X, we think of $X(*)$ as the underlying set of X and about $X(S)$ as the continuous maps from S to X. A quasi-separated[13] condensed set X is trivial as soon as $X(*)$ is trivial (a fact that is false for a general X, see [9, Lecture I]).

Let Top denote the category of T1 topological spaces[14]. We have a functor

$$\underline{(\ )} : \mathrm{Top} \to \mathrm{Cond}, \quad T \mapsto \underline{T} = \mathscr{C}(S,T),$$

where $\mathscr{C}(S,T)$ denotes the set of continuous functions from S to T. Condensed sets that come from topological spaces we will call *classical*. We quote:

Proposition 2.10. (Clausen-Scholze, [9, Prop. 1.2]). *The functor $\underline{(\ )}$:*

(1) has a left adjoint $X \mapsto X(*)_{\mathrm{top}}$ sending any condensed set X to the set $X(*)$ equipped with the quotient topology arising from the map

$$\coprod_{S, a \in X(S)} S \xrightarrow{a} X(*).$$

(2) restricted to compactly generated topological spaces is fully faithful.
(3) induces an equivalence between the category of compact Hausdorff spaces and qcqs (i.e., quasi-compact quasi-separated) condensed sets.

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[12] We refer the reader to [8, Lecture III] for a discussion of set theoretical issues involved in this definition.

[13] A condensed set X is called *quasi-compact* if there is a profinite set S and a surjective map $S \to X$; it is called *quasi-separated* if, for any pair of profinite sets S and S' over X the fiber product $S \times_X S'$ is quasi-compact.

[14] A topological space is called T1 if all its points are closed.

(4) induces a fully faithful functor from the category of compactly generated weak Hausdorff[15] *spaces, to quasi-separated condensed sets. The category of quasi-separated condensed sets is equivalent to the category of ind-compact Hausdorff spaces "*colim_n*"*X_n*, where all transition maps are closed immersions. If* $\{X_n\}_{n\in\mathbf{N}}$ *is a ind-system of compact Hausdorff spaces with closed immersions and* $X = \mathrm{colim}_n X_n$ *as topological spaces, then the canonical map*

$$\mathrm{colim}_n \underline{X_n} \to \underline{X}$$

is an isomorphism of condensed sets. In particular, $\mathrm{colim}_n \underline{X_n}$ *is classical, i.e., it comes from a topological space.*

Remark 2.11. (1) For $T \in \mathrm{Top}$, the counit $\underline{T}(*)_{\mathrm{top}} \to T$ of the adjunction agrees with the counit $T^{\mathrm{cg}} \to T$ of the adjunction between compactly generated spaces and all topological spaces by [8, Prop. 1.7]. In particular, $\underline{T}(*)_{\mathrm{top}} \simeq T^{\mathrm{cg}}$.

(2) For a finite extension K of $\mathbf{Q}_p$, we will abbreviate the notation $\mathrm{Mod}^{\mathrm{cond}}_{\underline{K}}$ to $\mathrm{Mod}^{\mathrm{cond}}_K$; we will call the elements of this category *condensed K-vector spaces*.

2.2.2. *Solid modules.* We will briefly review basic facts concerning solid modules.

(•) *Analytic rings.* We start with analytic rings.

(*i*) (Clausen-Scholze, [8, Th. 5.8]) The analytic ring $\mathbf{Z}_\square = (\mathbf{Z}, \mathscr{M}_{\mathbf{Z}})$ is defined as the ring $\mathbf{Z}$ equipped with the functor of measures $\mathscr{M}_{\mathbf{Z}}$ sending an extremally disconnected set $S = \lim_i S_i$, where each S_i is a finite set, to the condensed abelian group $\mathbf{Z}_\square := \lim_i \mathbf{Z}[S_i]$.

(*ii*) (Clausen-Scholze, [8, Prop. 7.9]) Let K be a finite extension of $\mathbf{Q}_p$. There is an analytic structure on the condensed rings $\mathscr{O}_K$ and K given by sending an extremally disconnected set $S = \lim_i S_i$, where each S_i is finite, to

$$\mathscr{O}_{K,\square}[S] := \lim_i \mathscr{O}_K[S_i], \quad K_\square[S] := K \otimes_{\mathscr{O}_K} \mathscr{O}_{K,\square}[S].$$

The first analytic ring structure is induced from the analytic ring structure of $\mathbf{Z}_\square$ by base change to $\mathscr{O}_K$.

(•) *Solid modules.* Now we pass to solid modules.

Proposition 2.12. (Clausen-Scholze [8, Prop. 7.5]) *Let $A = (A, \mathscr{M}_A)$ be one of the analytic rings above.*

(1) The full subcategory of solid A-modules

$$\mathrm{Mod}^{\mathrm{solid}}_A \subset \mathrm{Mod}^{\mathrm{cond}}_A \tag{2.13}$$

consists of all A-modules M such that, for all extremally disconnected sets S, the maps

$$\mathrm{Hom}_A(\mathscr{M}_A[S], M) \to \mathrm{Hom}_A(A[S], M)$$

are isomorphisms. It is an abelian subcategory, stable under all limits, colimits, and extensions. The inclusion (2.13) *admits a left adjoint*

$$\mathrm{Mod}^{\mathrm{cond}}_A \to \mathrm{Mod}^{\mathrm{solid}}_A : \quad M \mapsto M \otimes_A (A, \mathscr{M}_A), \tag{2.14}$$

which preserves all colimits and is symmetric monoidal.

(2) The functor

$$\mathscr{D}(\mathrm{Mod}^{\mathrm{solid}}_A) \to \mathscr{D}(\mathrm{Mod}^{\mathrm{cond}}_A) \tag{2.15}$$

is fully faithful. Its essential image is stable under all limits and colimits. It is given by complexes $M \in \mathscr{D}(\mathrm{Mod}^{\mathrm{cond}}_A)$ such that the map

$$\mathrm{RHom}_A(\mathscr{M}_A[S], M) \to \mathrm{RHom}_A(A[S], M)$$

[15] A topological space is called *weak Hausdorff* if the image of every continuous map from a compact Hausdorff space into the space is closed. In particular, every Hausdorff space is weak Hausdorff. Every weak Hausdorff space is a T1 space.

is a quasi-isomorphism for all extremally disconnected sets S.

A complex $M \in \mathscr{D}(\mathrm{Mod}_A^{\mathrm{cond}})$ is in $\mathscr{D}(\mathrm{Mod}_A^{\mathrm{solid}})$ if and only if $H^i(M)$ is in $\mathrm{Mod}_A^{\mathrm{solid}}$, for all i. The functor (2.15) admits a left adjoint

$$\mathscr{D}(\mathrm{Mod}_A^{\mathrm{cond}}) \to \mathscr{D}(\mathrm{Mod}_A^{\mathrm{solid}}) : M \mapsto M \otimes^{\mathrm{L}}_A (A, \mathscr{M}_A), \tag{2.16}$$

which is the left derived functor of (2.14). It is symmetric monoidal.

(3) For $M, N \in \mathscr{D}(\mathrm{Mod}_A^{\mathrm{solid}})$, we have the derived internal Hom

$$\mathrm{R}\underline{\mathrm{Hom}}_A(M,N) \in \mathscr{D}(\mathrm{Mod}_A^{\mathrm{solid}}).$$

The natural map $\mathrm{R}\underline{\mathrm{Hom}}_{(A,\mathscr{M}_A)}(M,N) \to \mathrm{R}\underline{\mathrm{Hom}}_A(M,N)$ is a quasi-isomorphism.

Notation 2.17. (1) We write Solid $:= \mathrm{Mod}^{\mathrm{solid}}_{(\mathbf{Z},\mathscr{M}_{\mathbf{Z}})}$ for the category of solid abelian groups; we write $\mathrm{CondAb} \to \mathrm{Solid} : M \mapsto M^{\square}$ for the functor (5.22) of Proposition 2.12, and call it *solidification*, and we denote by $\otimes^{\square}_{\mathbf{Z}}$ the unique symmetric monoidal tensor product making the solidification functor symmetric monoidal.

(2) For a finite extension K of $\mathbf{Q}_p$, we write $\mathrm{Mod}^{\mathrm{solid}}_{\mathscr{O}_K}$ and $\mathrm{Mod}^{\mathrm{solid}}_K$ for the categories of solid $\mathscr{O}_K$-modules and K-vector spaces, respectively. We will denote by $\mathscr{D}(\mathscr{O}_{K,\square})$ and $\mathscr{D}(K_\square)$ the corresponding derived ∞-categories.

(3) For a finite extension K of $\mathbf{Q}_p$ and a commutative solid K-algebra A, we write $\otimes^{\square}_A$ for the symmetric monoidal tensor product $\otimes_{(A,\mathscr{M}_A)}$.

2.2.3. *Locally convex and condensed vector spaces.* Consider the functor

$$\mathrm{CD} := \underline{(\ )} : \quad C_K^{\mathrm{Hcg}} \to \mathrm{Mod}_K^{\mathrm{cond}}, \quad V \mapsto \underline{V}.$$

We will denote in the same way its extension to the category of complexes. By Lemma 2.18 below, the functor CD preserves (strict) quasi-isomorphisms of complexes of Fréchet spaces or spaces of compact type and if V is such a complex then

$$\mathrm{CD}(\widetilde{H}^i(V)) \simeq H^i(\mathrm{CD}(V)).$$

Lemma 2.18. *The functor $\underline{(\ )}$ maps strict exact sequences of Fréchet spaces over K to exact sequences of condensed K-vector spaces. Similarly, for strict exact sequences of spaces of compact type over K.*

Proof. Since the functor $V \mapsto \underline{V}$ is left exact, it suffices to show that a strict surjection $V \to W$ of Fréchet spaces or of spaces of compact type is carried to a surjection $\underline{V} \to \underline{W}$. For that, it suffices to show that, for an extremally disconnected set S, we have $\mathscr{C}(S,V) \twoheadrightarrow \mathscr{C}(S,W)$, i.e., given $g \in \mathscr{C}(S,W)$, there exists $g' \in \mathscr{C}(S,V)$ making the following diagram commute

$$\begin{tikzcd} & V \arrow[d, "f"] \\ S \arrow[r, "g"'] \arrow[ur, dashed, "g'"] & W \end{tikzcd}$$

Assume first that both V and W are Fréchet. In that case we have the following argument of Guido Bosco [5, Lemma 1.A.33]: Since $g(S)$ is compact in W, by [33, Lemma 45.1], it is the image $f(H)$ of a compact subset H of V. We conclude by recalling that the extremally disconnected sets are the projective objects of the category of compact Hausdorff topological spaces.

Assume now that both V and W are of compact type. Since $g(S)$ is compact in W, if we write $W = \mathrm{colim}_n W_n$, for an ind-system $\{W_n\}_{n\in\mathbf{N}}$ of Banach spaces over K with compact injective transition maps, then $g(S) \subset W_m$, for some $m \in \mathbf{N}$ (see [30, Lemma 16.9]). By [29, Lemma 3.39], we have a commutative diagram of solid arrows

$$\begin{tikzcd} V_m \arrow[r, two heads, "f_m"] \arrow[d] & W_m \arrow[d, "g'"] & \\ V \arrow[r, two heads, "f"] & W & S, \arrow[l, "g"] \arrow[ul, "g_{W_m}"'] \arrow[ull, dashed] \end{tikzcd}$$

for a K- Banach space V_m. As above, it follows that the dashed arrow exists, which concludes our proof of the lemma. □

2.2.4. *Solid Fréchet spaces and solid spaces of compact type.* We will define solid Fréchet spaces and solid spaces of compact type as the images in the category of solid K-vector spaces of their classical analogs. See [29, Ch. 3] for alternative definitions.

(i) *Solid Fréchet spaces.* A *solid Banach space* is a solid K-vector space of the form $\underline{V}$, for a classical Banach space V. A *solid Fréchet space* is a solid K-vector space of the form $\underline{V}$, for a classical Fréchet space V. It is called *nuclear* if V is nuclear. The functor $(\underline{})$ identifies the categories of classical and solid Fréchet spaces (identifying also exact sequences by Lemma 2.18). If this does no confusion we will use the term "Fréchet spaces" for elements of both of these categories.

Warning. The definition of nuclear used here is stronger than the definition of nuclear in the solid formalism. To distinguish, we will call the latter "solid nuclear".

Let V be a solid Fréchet space written as a limit $V = \lim_n V_n$ of Banach spaces V_n with dense transition maps. Then:

(1) (Topological Mittag-Leffler) Then
$$\mathrm{R}^j \lim_n V_n = 0, \quad j \geq 1.$$
In particular, $V \simeq \mathrm{R}\underline{\mathrm{Hom}}_K(V^*, K)$.

(2) If W is a Banach space, then $\underline{\mathrm{Hom}}_K(V, W) = \mathrm{colim}_n\, \underline{\mathrm{Hom}}_K(V_n, W)$.

(ii) *Solid spaces of compact type.* A *solid space of compact type* is a solid K-vector space of the form $\underline{V}$, for a classical space of compact type V. The functor $(\underline{})$ identifies the categories of classical and solid spaces of compact type (identifying also exact sequences by Lemma 2.18). If this does no confusion we will use the term "spaces of compact type" for elements of both of these categories.

The following two lemmas will be needed later.

Lemma 2.19. *Let $V \in C_K$ be of compact type. Write it as $V \simeq \mathrm{colim}_n V_n$ with V_n, $n \in \mathbf{N}$, Hausdorff and injective compact transition maps. Then we have a canonical isomorphism of condensed K-modules*
$$\mathrm{colim}_n\, \underline{V_n} \xrightarrow{\sim} \underline{V}.$$
In particular, the condensed K-module $\mathrm{colim}_n\, \underline{V_n}$ is classical.

Proof. Since compact sets are bounded, this follows immediately from [30, Lemma 16.9], which states that any bounded subset of V comes from some V_n. □

Lemma 2.20. *If a sequence of solid Fréchet spaces over K is exact then the classical sequence is strictly exact. Similarly, for a sequence of solid spaces of compact type over K.*

Proof. We start with an exact sequence of solid Fréchet spaces over K
$$0 \to \underline{V}_1 \to \underline{V}_2 \to \underline{V}_3 \to 0.$$
We need to show that the sequence of classical Fréchet spaces
$$0 \to V_1 \xrightarrow{f_1} V_2 \to V_3 \to 0 \tag{2.21}$$

is strictly exact. But the sequence

$$0 \to \underline{V}_1(*)_{\rm top} \to \underline{V}_2(*)_{\rm top} \to \underline{V}_3(*)_{\rm top} \to 0 \tag{2.22}$$

maps, via topological isomorphisms $\underline{V}_i(*)_{\rm top} \xrightarrow{\sim} V_i$ (since V_i is compactly generated), to the sequence (2.21). Since $\underline{V_3}$ is quasi-separated, the map f_1 is a closed immersion. In particular we have a strict exact sequence of Fréchet spaces

$$0 \to V_1 \xrightarrow{f_1} V_2 \to V_3' \to 0$$

and a continuous injection $f_2 : V_3' \to V_3$. We need to prove that the map f_2 is a topological isomorphism. Since f_2 is a map between two Fréchet spaces, by the Open Mapping Theorem, it suffices to show that f_2 is an algebraic isomorphism. But, by Lemma 2.18, we have the exact sequence

$$0 \to \underline{V}_1 \to \underline{V}_2 \to \underline{V}_3' \to 0.$$

Hence the canonical map $\underline{f}_2 : \underline{V}_3' \to \underline{V}_3$ is an isomorphism, which yields that so is the map f_3 (by the faithfullness of the (_) functor.

The argument for spaces of compact type is similar (the Open Mapping Theorem is valid for LB spaces). □

2.2.5. *Solid tensor product.* We list properties of the solid tensor product that we will often use.

(1) ([5, Prop. A.68]) Let V, W be Fréchet spaces over K. Then we have a natural isomorphism of solid K-vector spaces

$$\underline{V} \otimes^{\square}_K \underline{W} \xrightarrow{\sim} \underline{V \widehat{\otimes}_K W},$$

where $V\widehat{\otimes}_K W$ denotes the projective tensor product in the category C_K.

(2) ([5, Cor. A.65]) Any Fréchet space over K is acyclic for the tensor product $\otimes^{\square}_K$. That is, if V is a Fréchet space over K then $(-) \otimes^{\mathrm{L}\square}_K V \simeq (-) \otimes^{\square}_K V$.

(3) ([5, Cor. A.67])
   (a) Let $\{V_n\}_{n\in\mathbf{N}}$ be a pro-system of solid nuclear K-vector spaces and let W be a Fréchet vector space over K. Then we have an isomorphism

$$\lim_n (V_n \otimes^{\square}_K W) \xleftarrow{\sim} (\lim_n V_n) \otimes^{\square}_K W.$$

   (b) Let $\{V_n\}_{n\in\mathbf{N}}$ be a pro-system in $\mathscr{D}(K_\square)$ of complexes of solid nuclear K-vector spaces. Let W be a complex of K-Fréchet spaces. Then we have a quasi-isomorphism

$$\mathrm{R}\lim_n (V_n \otimes^{\mathrm{L}\square}_K W) \xleftarrow{\sim} (\mathrm{R}\lim_n V_n) \otimes^{\mathrm{L}\square}_K W.$$

The above properties also hold if we replace K with $\breve{F}$ (with the same references).

3. Galois cohomology of K

This chapter gathers together a number of properties of Galois cohomology that we will need later.

3.1. **Preliminaries.** We record here few basic facts about Galois cohomology seen via condensed formalism.

3.1.1. *Condensed Galois cohomology.* Let G be a condensed group. A condense G-module is a condensed abelian group endowed with a $\mathbf{Z}[G]$-module structure. The condensed group cohomology of G with values in a condense G-module V is defined as

$$\mathrm{R}\Gamma(G, V) := \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{Z}[G]}(\mathbf{Z}, V) \in \mathscr{D}(\mathrm{CondAb}).$$

Notation 3.1. Let G be a profinite group.

(a) The *Iwasawa* algebra of G is the solid ring

$$\mathscr{O}_{K,\square}[G] := \lim_{H\subset G} \mathscr{O}_K[G/H] \in \mathrm{Mod}^{\mathrm{solid}}_{\mathscr{O}_K},$$
$$K_\square[G] = \mathscr{O}_{K,\square}[G][1/p] := \big(\lim_{H\subset G} \mathscr{O}_K[G/H]\big)[1/p] \in \mathrm{Mod}^{\mathrm{solid}}_K,$$

where H runs over all open and normal subgroups of G.

(b) A solid G-module over $\mathscr{O}_K$ (or a solid $\mathscr{O}_{K,\square}[G]$-module) is a solid abelian group endowed with an $\mathscr{O}_{K,\square}[G]$-module structure. The category of solid $\mathscr{O}_{K,\square}[G]$-modules will be denoted by $\mathrm{Mod}^{\mathrm{solid}}_{\mathscr{O}_{K,\square}[G]}$ and its derived ∞-category by $\mathscr{D}(\mathscr{O}_{K,\square}[G])$. Similarly, for solid $K_\square[G]$-modules.

We list the following properties of $\mathrm{R}\Gamma(G,-)$:

(1) ([5, Prop. B.2]) Let G be a profinite group and let V be a G-module in solid abelian groups. Then
   (a) The complex $\mathrm{R}\Gamma(G,V)$ is quasi-isomorphic to the complex of solid abelian groups

$$V \to \underline{\mathrm{Hom}}(\mathbf{Z}[G^1],V) \to \underline{\mathrm{Hom}}(\mathbf{Z}[G^2],V) \to \underline{\mathrm{Hom}}(\mathbf{Z}[G^3],V)\cdots. \tag{3.2}$$

   (b) If $V = \underline{V_{\mathrm{top}}}$, with V_{top} a T1 topological G-module over $\mathbf{Z}$, then, for all $i \geq 0$, we have a natural isomorphism of abelian groups[16]

$$\mathrm{R}\Gamma(G,V)(*) \simeq \mathrm{R}\Gamma(G,V_{\mathrm{top}}).$$

(2) ([5, Prop. B.3]) For $n \in \mathbf{N}$, let $\Gamma := \mathbf{Z}_p^n$, and let $\gamma_1,\ldots,\gamma_n$ denote the generators of Γ. Let V be a Γ-module in $\mathrm{Mod}^{\mathrm{solid}}_{\mathbf{Z}_p}$. Then we have a quasi-isomorphism

$$\mathrm{R}\Gamma(\Gamma,V) \simeq \mathrm{Kos}_\gamma(V) := \mathrm{Kos}_V(\gamma_1-1,\ldots,\gamma_n-1),$$

the Koszul complex of V with respect to the elements $\gamma_1-1,\ldots,\gamma_n-1$.

(3) ([29, Lemma 5.2]) Let G be a profinite group. There is a solid projective resolution of the trivial representation

$$\cdots \to K_\square[G^{n+1}] \to K_\square[G^n] \to \cdots \to K_\square[G^1] \to K \to 0.$$

In particular, if V is a G-module in solid modules over K, one has that

$$\mathrm{R}\underline{\mathrm{Hom}}_{K_\square[G]}(K,V) \simeq \mathrm{R}\Gamma(G,V).$$

Lemma 3.3. *Let G be a profinite group and let V be a finite rank $\mathbf{Q}_p$-vector space equipped with a continuous action of G. Then*

(1) we have a quasi-isomorphism and isomorphisms

$$\mathrm{CD}(\mathrm{R}\Gamma(G,V)) \simeq \mathrm{R}\Gamma(G,\underline{V}), \quad \mathrm{CD}(\widetilde{H}^i(G,V)) \simeq H^i(G,\underline{V}),\ i\geq 0.$$

(2) we have a quasi-isomorphism[17] in $\mathscr{D}(C_{\mathbf{Q}_p})$

$$\mathrm{R}\Gamma(G,\underline{V})(*)_{\mathrm{top}} \simeq \mathrm{R}\Gamma(G,V).$$

Proof. For claim (1) we compute

$$\mathrm{R}\Gamma(G,V) \simeq C(G,V), \quad n \mapsto \mathscr{C}(G^{n-1},V);$$
$$\mathrm{CD}(\mathrm{R}\Gamma(G,V))(S) : n \mapsto \mathscr{C}(S,\mathscr{C}(G^{n-1},V)) \simeq \mathscr{C}(S\times G^{n-1},V),$$

[16]The second cohomology group is the continuous group cohomology.

[17]Topology on $\mathrm{R}\Gamma(G,V)$ is defined using continuous cochains. See the proof of the lemma for details.

where S is a profinite set and $\mathscr{C}(-,-)$ denotes the space of continuous maps equipped with compact open topology (note that $\mathscr{C}(S\times G^{n-1},V)$, since $S\times G^{n-1}$ is compact, is a $\mathbf{Q}_p$-Banach space). We also have

$$\begin{aligned}&\mathrm{R}\Gamma(G,\underline{V}): n\mapsto \underline{\mathrm{Hom}}(\mathbf{Z}[G^{n-1}],\underline{V});\\ &\underline{\mathrm{Hom}}(\mathbf{Z}[G^{n-1}],\underline{V})(S)=\mathrm{Hom}(\mathbf{Z}[S]\otimes\mathbf{Z}[G^{n-1}],\underline{V})\simeq\mathrm{Hom}(\mathbf{Z}[S\times G^{n-1}],\underline{V}).\end{aligned}$$

Since $\mathrm{Hom}(\mathbf{Z}[S\times G^{n-1}],\underline{V})\simeq\mathscr{C}(S\times G^{n-1},V)$, we get claim (1) of the lemma.

Claim (2) follows from the fact that the complex of continuous cochains representing $\mathrm{R}\Gamma(G,V)$ is a complex of Banach spaces and [29, Prop. 3.5]. □

3.1.2. *Poitou-Tate duality.* Let K be a finite extension of $\mathbf{Q}_p$ and let V be a continuous, finite rank $\mathbf{Q}_p$-representation of $\mathscr{G}_K$. Recall that the Galois pairing

$$H^i(\mathscr{G}_K,V)\otimes^{\square}_{\mathbf{Q}_p}H^{2-i}(\mathscr{G}_K,V^*(1))\xrightarrow{\cup}H^2(\mathscr{G}_K,\mathbf{Q}_p(1))\underset{\sim}{\xrightarrow{\mathrm{Tr}_K}}\mathbf{Q}_p$$

is a perfect pairing (by Poitou-Tate duality). Hence, for $i\in\mathbf{N}$, $H^i(\mathscr{G}_K,V)$ and $H^{2-i}(\mathscr{G}_K,V^*(1))$ are natural duals (via the above pairing). In particular, we have

$$H^0(\mathscr{G}_K,\mathbf{Q}_p(j))\simeq\begin{cases}\mathbf{Q}_p & \text{if } j=0,\\ 0 & \text{otherwise;}\end{cases}\qquad H^2(\mathscr{G}_K,\mathbf{Q}_p(j))\simeq\begin{cases}\mathbf{Q}_p & \text{if } j=1,\\ 0 & \text{otherwise.}\end{cases}$$

3.2. **(φ,Γ)-modules and Galois cohomology.** In the next two sections, we will briefly recall and refine the relationship between (φ,Γ)-modules and Galois cohomology.

3.2.1. *Notations.* If $n\geq 1$, let $F_n=\mathbf{Q}_p(\boldsymbol{\mu}_{p^n})$ and let $F_\infty:=\cup_n F_n$ be the cyclotomic extension of $\mathbf{Q}_p$. Let $\chi:\mathscr{G}_{\mathbf{Q}_p}\to\mathbf{Z}_p^*$ be the cyclotomic character. Then χ factors through $\Gamma:=\mathrm{Gal}(F_\infty/\mathbf{Q}_p)$ and induces an isomorphism $\chi:\Gamma\xrightarrow{\sim}\mathbf{Z}_p^*$.

If Δ is the torsion subgroup of Γ, then $\chi^{|\Delta|}$ takes values in $1+p\mathbf{Z}_p$ (resp. $1+8\mathbf{Z}_2$) if $p\neq 2$ (resp. $p=2$). Let $\tau:\mathscr{G}_{\mathbf{Q}_p}\to\mathbf{Z}_p$ be defined by

$$\tau=\begin{cases}\frac{1}{p\,|\Delta|}\log\chi^{|\Delta|} & \text{if } p\neq 2,\\ \frac{1}{4\,|\Delta|}\log\chi^{|\Delta|} & \text{if } p=2.\end{cases}\quad\text{i.e., }\tau=\tfrac{1}{p^{c(p)}}\log\chi,\ \text{with } c(p)=\begin{cases}1 & \text{if } p\neq 2,\\ 2 & \text{if } p=2.\end{cases}$$

Let $F'_\infty:=F_\infty^\Delta$ be the cylotomic $\mathbf{Z}_p$-extension of $\mathbf{Q}_p$. Then τ factors through $\Gamma':=\mathrm{Gal}(F'_\infty/\mathbf{Q}_p)$ and induces an isomorphism $\tau:\Gamma'\xrightarrow{\sim}\mathbf{Z}_p$.

Let K be a finite extension of $\mathbf{Q}_p$. If $n\in\mathbf{N}$, let $K_n=K(\boldsymbol{\mu}_{p^n})$ and let $K_\infty:=\cup_n K_n$ be the cyclotomic extension of K. Let $\Gamma_K:=\mathrm{Gal}(K_\infty/K)$; then χ induces an isomorphism from Γ_K to an open subgroup of $\mathbf{Z}_p^*$. Let Δ_K be the torsion subgroup of Γ_K, let $K'_\infty:=K_\infty^{\Delta_K}$ be the cylotomic $\mathbf{Z}_p$-extension of K. Then $\Gamma'_K:=\mathrm{Gal}(K'_\infty/K)=\Gamma_K/\Delta_K$ and τ induces an isomorphism $\tau:\Gamma'_K\xrightarrow{\sim}p^{n(K)}\mathbf{Z}_p$ for some $n(K)\in\mathbf{N}$.

Let $\gamma_K\in\Gamma'_K$ be the element verifying $\tau(\gamma_K)=p^{n(K)}$. Then γ_K has a unique lifting in Γ_K whose image by χ belongs to $1+p^{n(K)+c(p)}\mathbf{Z}_p$; we denote this lifting also by γ_K. Then $\Gamma_K=\gamma_K^{\mathbf{Z}_p}\times\Delta_K$.

Let $L=K(\boldsymbol{\mu}_{p^{c(p)}})$. Then $\mathrm{Gal}(L/K)=\Delta_K$ and $\Gamma_L=\gamma_K^{\mathbf{Z}_p}$.

Remark 3.4. (i) Let $F:=K\cap F_\infty$. Then $[K_\infty:F_\infty]=[K:F]$ and $\Gamma_F=\Gamma_K=\Delta_K\times p^{n(K)}\mathbf{Z}_p$; hence $[F:\mathbf{Q}_p]=p^{n(K)}\frac{|\Delta|}{|\Delta_K|}$ and $[K:\mathbf{Q}_p]=[K_\infty:F_\infty]\cdot p^{n(K)}\frac{|\Delta|}{|\Delta_K|}$.

(ii) We have $\tau(\gamma_K)=p^{n(K)}$, hence $\log\chi(\gamma_K)=p^{n(K)+c(p)}$.

For $0<u\leq v\in v(K_\infty^\flat)$, let

$$\mathbf{B}_{K_\infty}^{[u,v]}:=(W(\mathscr{O}_{K_\infty}^\flat)[\tfrac{p}{[\alpha]},\tfrac{[\beta]}{p}]^{\wedge_p})[\tfrac{1}{p}],\quad v(\alpha)=\tfrac{1}{v},\ v(\beta)=\tfrac{1}{u}.$$

It is a Banach space over $\mathbf{Q}_p$. Let φ denote the Frobenius morphism acting on $\mathbf{B}_{K_\infty}^{[u,v]}$ and ψ its left inverse. Let $U^{[u,v]}$ be the corresponding open set of the Fargues-Fontaine curve over K_∞. Take

$u=\frac{p-1}{p}$, $v=p-1$ if $p\neq 2$, and $u=\frac{2}{3}$, $v=\frac{4}{3}$ if $p=2$. Then $\mathbf{B}^{[u,v]}_{K_\infty}/t=\widehat{K}_\infty$ and t is a unit in [18] $\mathbf{B}^{[u,v/p]}_{K_\infty}$. We write θ the canonical map $\mathbf{B}^{[u,v]}_{K_\infty}\to\widehat{K}_\infty$ with $\mathrm{Ker}(\theta)=t\mathbf{B}^{[u,v]}_{K_\infty}$.

3.2.2. *Galois cohomology.* For $j\in\mathbf{Z}$, the theory of (φ,Γ)-modules yields quasi-isomorphisms

$$\begin{aligned}\alpha_j&:\mathrm{Kos}_{\varphi,\gamma}(\mathbf{B}^{[u,v]}_{K_\infty}(j))\simeq \mathrm{R}\Gamma(\mathscr{G}_L,\mathbf{Q}_p(j)),\\ \alpha_j&:\mathrm{Kos}_{\varphi,\gamma}(\mathbf{B}^{[u,v]}_{K_\infty}(j))^{\Delta_K}\simeq \mathrm{R}\Gamma(\mathscr{G}_K,\mathbf{Q}_p(j)),\end{aligned}\tag{3.5}$$

where the (φ,γ)-Koszul complexes (of Banach spaces over $\mathbf{Q}_p$) are defined by:

$$\mathrm{Kos}_{\varphi,\gamma}(\mathbf{B}^{[u,v]}_{K_\infty}(j)):=\ \big((\mathbf{B}^{[u,v]}_{K_\infty}(j))\xrightarrow{(\varphi-1,\gamma_K-1)}(\mathbf{B}^{[u,v/p]}_{K_\infty}(j))\oplus(\mathbf{B}^{[u,v]}_{K_\infty}(j))\xrightarrow{-(\gamma_K-1)+(\varphi-1)}(\mathbf{B}^{[u,v/p]}_{K_\infty}(j))\big)$$

and the complex $\mathrm{Kos}_{\varphi,\gamma}(\mathbf{B}^{[u,v]}_{K_\infty}(j))^{\Delta_K}$ is obtained by taking fixed points under Δ_K of each of the terms of the complex.

Set $[\Delta_K]:=\sum_{\sigma\in\Delta_K}\sigma$. The following commutative diagram allows to deduce results for K from results for L, i.e. we can often assume that $\Delta_K=1$ in the proofs,

$$\begin{tikzcd}
\mathrm{Kos}_{\varphi,\gamma}(\mathbf{B}^{[u,v]}_{K_\infty}(j)) \arrow[r, "\sim"] \arrow[d, shift left, "{[\Delta_K]}"] & \mathrm{R}\Gamma(\mathscr{G}_L,\mathbf{Q}_p(j)) \arrow[d, shift left, "\mathrm{cor}_L^K"] \\
\mathrm{Kos}_{\varphi,\gamma}(\mathbf{B}^{[u,v]}_{K_\infty}(j))^{\Delta_K} \arrow[r, "\sim"] \arrow[u, shift left, "\mathrm{id}"] & \mathrm{R}\Gamma(\mathscr{G}_K,\mathbf{Q}_p(j)) \arrow[u, shift left, "\mathrm{res}_K^L"]
\end{tikzcd}$$

Remark 3.6. If n is big enough so that K_n has enough roots of unity in the sense of [16], there exist normalized trace maps $\mathrm{Res}_{p^{-n}\mathbf{Z}_p}:\mathbf{B}^{[u,v]}_{K_\infty}\to\mathbf{B}^{[u,v]}_{K_n}$ where $\mathbf{B}^{[u,v]}_{K_n}:=\varphi^{-n}(\mathbf{B}_K^{[u/p^n,v/p^n]})$ which commute with Γ_K and verify $\varphi\circ\mathrm{Res}_{p^{-n-1}\mathbf{Z}_p}=\mathrm{Res}_{p^{-n}\mathbf{Z}_p}\circ\varphi$. These decompletion maps play a big role in the proofs because $\mathbf{B}^{[u,v]}_{K_n}$ is a much nicer ring than $\mathbf{B}^{[u,v]}_{K_\infty}$ (it is a ring of analytic functions on an annulus, whereas the later is a ring of analytic functions on a perfectoid annulus).

Applying $\mathrm{Res}_{p^{-n}\mathbf{Z}_p}$ to each of the terms of the above complex produces a quasi-isomorphic complex with rings related to K_n instead of K_∞, which is closer to the complexes used in the theory of (φ,Γ)-modules, like in Herr's thesis [23] or in [6]. Going from these rings to the usual rings uses the standard techniques of (φ,Γ)-modules, i.e., normalized trace $\mathrm{Res}_{p^{-n}\mathbf{Z}_p}$, bijectivity of γ_K-1 on $(\mathbf{B}_K^{[u/p^n,v/p^n]}(j))^{\psi=0}$, etc., as in [16].

3.2.3. *Examples.* Denote by

$$h^i_K: Z^i(\mathrm{Kos}_{\varphi,\gamma}(\mathbf{B}^{[u,v]}_{K_\infty}(j))^{\Delta_K})\to H^i(\mathscr{G}_K,\mathbf{Q}_p(j))\tag{3.7}$$

the map induced by quasi-isomorphism (3.5); it factorizes as

$$h^i_K: Z^i(\mathrm{Kos}_{\varphi,\gamma}(\mathbf{B}^{[u,v]}_{K_\infty}(j))^{\Delta_K})\to H^i(\mathrm{Kos}_{\varphi,\gamma}(\mathbf{B}^{[u,v]}_{K_\infty}(j))^{\Delta_K})\simeq H^i(\mathscr{G}_K,\mathbf{Q}_p(j)).$$

If $(a,b)\in Z^1(\mathrm{Kos}_{\varphi,\gamma}(\mathbf{B}^{[u,v]}_{K_\infty}(j))^{\Delta_K})$ and $u\in\mathbf{B}^{[u,v]}_{\overline{K}}(j)$ satisfy $(\varphi-1)u=a$, then

$$h^1_K(a,b)=\mathrm{cl}\big(\sigma\mapsto\tfrac{\sigma-1}{\gamma_K-1}b-(\sigma-1)u\big).\tag{3.8}$$

(In this expression, σ acts through its image in Γ'_K on b, and we think of $\frac{\sigma-1}{\gamma_K-1}$ as an element of $\mathbf{Z}_p[[\Gamma'_K]]$; the expression between parenthesis is a 1-cocycle on $\mathscr{G}_K$ with values in $\mathbf{Q}_p(j)$, and cl denotes its image in $H^1(\mathscr{G}_K,\mathbf{Q}_p(j))$.)

• *The case* $j=0$. We can apply the formula (3.8) to $j=0$ and cocycle $(1,0)$. Then the corresponding $\mathscr{G}_K$-cocycle factors through $\mathrm{Gal}(\overline{\mathbf{F}}_p/k_K)$ since the solution of $(\varphi-1)u=1$ belongs to $W(\overline{\mathbf{F}}_p)$, and sends the relative Frobenius to $f:=f(K/\mathbf{Q}_p)$ (because $(\sigma-1)u$ is equal to $(\varphi^f-1)u=f$ since $\varphi(u)=u+1$). Since the relative Frobenius corresponds to a uniformizer of

[18] That is, the intersection of $U^{[u,v]}$ with the orbit under φ of the point ∞ of the Fargues-Fontaine curve is reduced to $\{\infty\}$ and $U^{[u,v/p]}$ does not intersect this orbit.

K^*, it follows that the element λ of $\mathrm{Hom}(K^*, \mathbf{Q}_p)$ which corresponds to $h^1_K(1,0)$ is $f(K/\mathbf{Q}_p)v_K = v_p \circ \mathrm{N}_{K/\mathbf{Q}_p}$ (where v_K is the valuation on K with image $\mathbf{Z}$).

Starting with $(0, p^{n(K)+c(p)})$, formula (3.8) with $u = 0$ yields a group morphism $\mathscr{G}_K \to \mathbf{Z}_p$ which factors through Γ'_K and has value $\log \chi(\gamma_K)$ at γ_K; it follows that this morphism is $\log \chi$.

• *The case* $j = 1$. Let $n = n(K) + c(p)$. We have $(\varphi - 1)\frac{1}{\pi} \in (\mathbf{B}^{(0,v]}_{\mathbf{Q}_p})^{\psi=0}$ and there exists $a \in (\mathbf{B}^{(0,v/p^n]}_{\mathbf{Q}_p})^{\psi=0} \otimes \frac{d\pi}{1+\pi}$ such that $(\gamma_K - 1)a = ((\varphi - 1)\frac{1}{\pi}) \otimes \frac{d\pi}{1+\pi}$. Then,

$$(\varphi^{-n}(a), \varphi^{-n}(\tfrac{1}{\pi} \otimes \tfrac{d\pi}{1+\pi})) \in Z^1(\mathrm{Kos}_{\varphi,\gamma}(\mathbf{B}^{[u,v]}_{F_n}(1))) \quad \text{and} \quad h^1_{F_n}(\varphi^{-n}(a), \varphi^{-n}(\tfrac{1}{\pi} \otimes \tfrac{d\pi}{1+\pi})) = \mathrm{cl}(\zeta_{p^n} - 1).$$

(This follows from point iii) of [6, Prop. V.3.2], using the constructions leading to [6, th. II.1.3] and the (obvious) fact that Coleman power series attached to $(\zeta_{p^n} - 1)_{n \geq 1}$ is just T.)

3.2.4. *Cup-products.* We define compatible cup products ($\alpha \in \mathbf{Q}_p$)

$$\cup_\alpha : \quad \mathrm{Kos}_{\varphi,\gamma}(\mathbf{B}^{[u,v]}_{K_\infty}(j_1))^{\Delta_K} \otimes^{\Box}_{\mathbf{Q}_p} \mathrm{Kos}_{\varphi,\gamma}(\mathbf{B}^{[u,v]}_{K_\infty}(j_2))^{\Delta_K} \to \mathrm{Kos}_{\varphi,\gamma}(\mathbf{B}^{[u,v]}_{K_\infty}(j_1 + j_2))^{\Delta_K}$$

using (twice) the formulas from Section 7.3.2.

Lemma 3.9. *On the level of cohomology, the quasi-isomorphism* (3.5) *is compatible with products.*

Proof. This is easy to check for the pairing of 0- and 2-cocycles. For the pairing of two 1-cocycles, this amounts to checking that

$$h^1_K(a,b) \cup h^1_K(a',b') = h^2_K(b \otimes \gamma_K(a') - a \otimes \varphi(b')). \tag{3.10}$$

(For the product on the Koszul complexes, we used here (twice) the formulas from Section 7.3.2 with $\alpha = 1$.) But equality (3.10) is standard and was checked by Herr and Benois in [23, 3]. □

3.3. **Residues and duality.**

3.3.1. *The trace map.* We will describe the trace isomorphism

$$\mathrm{Tr}_K : H^2(\mathscr{G}_K, \mathbf{Q}_p(1)) \xrightarrow{\sim} \mathbf{Q}_p$$

in terms of (φ, Γ)-modules (cf. [23], [3]). For this, we start with the description of Tr_K given by local class field theory. We have isomorphisms

$$\mathrm{cl} : \mathrm{Hom}(K^*, \mathbf{Q}_p) \xrightarrow{\sim} H^1(\mathscr{G}_K, \mathbf{Q}_p) \quad \text{and} \quad \mathbf{Q}_p \widehat{\otimes} K^* \xrightarrow{\sim} H^1(\mathscr{G}_K, \mathbf{Q}_p(1))$$

and Tr_K is given by the formula

$$\mathrm{Tr}_K(\mathrm{cl}(\tau) \cup \mathrm{cl}(\alpha)) = \tau(\alpha), \quad \text{if } \tau \in \mathrm{Hom}(K^*, \mathbf{Q}_p) \text{ and } \alpha \in K^*.$$

3.3.2. *Trace map and residues.* Let $\pi := [\varepsilon] - 1 \in \mathbf{B}^{[u,v]}_{F_\infty}$. We have $\gamma(\pi) = (1+\pi)^{\chi(\gamma)} - 1$ if $\gamma \in \Gamma$, and we make φ and Γ act on $\frac{d\pi}{1+\pi}$ by $\varphi(\frac{d\pi}{1+\pi}) = \frac{d\pi}{1+\pi}$ and $\gamma(\frac{d\pi}{1+\pi}) = \chi(\gamma)\frac{d\pi}{1+\pi}$, respectively (this allows us to identify $\Lambda(1)$ with $\Lambda \otimes \frac{d\pi}{1+\pi}$). Then there exists a unique continuous linear map

$$\mathrm{res}_\pi : \mathbf{B}^{[u,v]}_{F_\infty} \otimes \tfrac{d\pi}{1+\pi} \to \mathbf{Q}_p$$

which commutes with φ and Γ, and sends $\sum_{k \in \mathbf{Z}} a_k \pi^k \, d\pi$ to a_{-1}. (See [13, Prop. IV.3.3] for the existence of res_π.)

Set

$$\mathrm{Tr}_{K_\infty/F_\infty} := \sum\nolimits_{\sigma \in G_{F_\infty}/G_{K_\infty}} \sigma$$

(well defined on modules on which $\mathscr{G}_{K_\infty}$ acts trivially).

Proposition 3.11. *We have*

$$\mathrm{Tr}_K \circ h^2_K = \tfrac{1}{|\Delta_K|} \mathrm{res}_\pi \circ \mathrm{Tr}_{K_\infty/F_\infty} \quad \text{on } (\mathbf{B}^{[u,v/p]}_{K_\infty} \otimes \tfrac{d\pi}{1+\pi})^{\Delta_K}$$

Proof. If $\alpha = \mathrm{res}_\pi \circ \mathrm{Tr}_{K_\infty/F_\infty}$, then $\alpha(\varphi(x)) = \alpha(x)$ and $\alpha(\sigma(x)) = \alpha(x)$, for all $\sigma \in \mathscr{G}_{\mathbf{Q}_p}$. In particular, α factors through $H^2(\mathrm{Kos}_{\varphi,\gamma}(\mathbf{B}_{K_\infty}^{[u,v]}(1)))$, which is of dimension 1 over $\mathbf{Q}_p$. This proves the result up to multiplication by a constant and to show that this constant is 1, it suffices to show that the two terms coincide on an element on which one of the two is nonzero.

Moreover, we have a commutative diagram

$$\begin{tikzcd}
\mathbf{B}_{K_\infty}^{[u,v/p]} \arrow[r, "h_L^2"] \arrow[d, "{[\Delta_K]}"] & H^2(\mathscr{G}_L, \mathbf{Q}_p(1)) \arrow[r, "\mathrm{Tr}_L"] \arrow[d, "\mathrm{cor}_{L/K}"] & \mathbf{Q}_p \arrow[d, equal] \\
(\mathbf{B}_{K_\infty}^{[u,v/p]})^{\Delta_K} \arrow[r, "h_K^2"] & H^2(\mathscr{G}_K, \mathbf{Q}_p(1)) \arrow[r, "\mathrm{Tr}_K"] & \mathbf{Q}_p
\end{tikzcd}$$

Hence $\mathrm{Tr}_L \circ h_L^2 = |\Delta_K| \,\mathrm{Tr}_K \circ h_K^2$ on $(\mathbf{B}_{K_\infty}^{[u,v/p]})^{\Delta_K}$. It follows that the result holds for K if and only if it holds for L, and we can assume that $\Delta_K = 1$, i.e. $K = K_n$, with $n = n(K) + c(p)$.

Let $\lambda = v_p \circ \mathrm{N}_{K/\mathbf{Q}_p}$ as in the case $j = 0$ in section 3.2.3. It follows, using the identity $\mathrm{N}_{F_n/\mathbf{Q}_p}(\zeta_{p^n} - 1) = p$ (resp. $= -2$ if $p = 2$), that

$$\begin{aligned}
\mathrm{cl}(\lambda) \cup \mathrm{cl}(\zeta_{p^n} - 1) &= v_p(\mathrm{N}_{K/\mathbf{Q}_p}(\zeta_{p^n} - 1)) \\
&= [K_n : F_n]\, v_p(\mathrm{N}_{F_n/\mathbf{Q}_p}(\zeta_{p^n} - 1)) = [K_n : F_n] = [K_\infty : F_\infty]
\end{aligned}$$

But formula (3.10) gives us

$$\mathrm{cl}(\lambda) \cup \mathrm{cl}(\zeta_{p^n} - 1) = \mathrm{Tr}_K \circ h_K^2\big(\varphi^{1-n}\big(\tfrac{1}{\pi} \otimes \tfrac{d\pi}{1+\pi}\big)\big)$$

Since $\mathrm{Tr}_{K_\infty/F_\infty}$ and $x \mapsto \mathrm{res}_\pi x \frac{d\pi}{1+\pi}$ commute with φ and Γ_K, we have

$$\mathrm{res}_\pi \circ \mathrm{Tr}_{K_\infty/F_\infty}\big(\varphi^{1-n}\big(\tfrac{1}{\pi} \otimes \tfrac{d\pi}{1+\pi}\big)\big) = [K_\infty : F_\infty]\, \mathrm{res}_\pi\big(\tfrac{1}{\pi} \otimes \tfrac{d\pi}{1+\pi}\big) = [K_\infty : F_\infty]$$

Our proposition follows. $\square$

3.3.3. *An alternative description of the trace map.* Let

$$\mathrm{Tr} : \overline{\mathbf{Q}}_p \to \mathbf{Q}_p$$

be the unique $\mathscr{G}_{\mathbf{Q}_p}$-equivariant projection (if $[L : \mathbf{Q}_p] < \infty$, the restriction of Tr to L coincides with $\frac{1}{[L:\mathbf{Q}_p]}\mathrm{Tr}_{L/\mathbf{Q}_p}$). Then Tr extends by continuity to $\widehat{K}_\infty$ (normalized Tate trace; it does not extend to $\mathbf{C}_p$), and $\mathrm{Tr} \circ \theta$ gives a well defined map $\mathbf{B}_{K_\infty}^{[u,v]} \to \mathbf{Q}_p$.

Proposition 3.12. *If $\alpha \in \mathbf{B}_{K_\infty}^{[u,v]}$,*

$$\mathrm{Tr}_K \circ h_K^2((\varphi - 1)\tfrac{\alpha}{t} \otimes \tfrac{d\pi}{1+\pi}) = -\tfrac{[K:\mathbf{Q}_p]}{\log \chi(\gamma_K)} \mathrm{Tr} \circ \theta(\alpha)$$

Proof. Denote by $\delta_K : \mathbf{B}_{K_\infty}^{[u,v]} \to \mathbf{Q}_p$ the map $\alpha \mapsto \delta_K(\alpha) := \mathrm{Tr}_K \circ h_K^2((\varphi - 1)\frac{\alpha}{t} \otimes \frac{d\pi}{1+\pi})$. Then δ_K is identically 0 on $\mathrm{Ker}\,\theta = t\mathbf{B}_{K_\infty}^{[u,v]}$ (because $(\varphi - 1)\frac{\alpha}{t}$ is then a coboundary), hence it factors through $\mathbf{B}_{K_\infty}^{[u,v]}/t = \widehat{K}_\infty$. It commutes with the action of $\mathscr{G}_{\mathbf{Q}_p}$ (i.e., $\delta_{\sigma(K)} \circ \sigma = \delta_K$, for all $\sigma \in \mathscr{G}_{\mathbf{Q}_p}$). So there exists $c(K)$ such that $\delta_K = c(K)\mathrm{Tr} \circ \theta$.

To determine $c(K)$, it is enough to compute the value of the term on the left-hand side for $\alpha = 1$, which can be done using Proposition 3.11. We have $(\varphi - 1)\frac{1}{t} = \frac{1-p}{p\,t}$, and $\mathrm{Tr}_{K_\infty/F_\infty}$ is multiplication by $[K_\infty : F_\infty]$ on $\mathbf{B}_{F_\infty}^{[u,v/p]}$ (which contains $\frac{1}{t}$). Finally, $\frac{1}{t}\frac{d\pi}{1+\pi} = (\frac{1}{\pi} + \sum_{n\geq 0} a_n \pi^n)\, d\pi$ in $\mathbf{B}_{\mathbf{Q}_p}^{[u,v/p]} \otimes \frac{d\pi}{1+\pi}$, and $\mathrm{res}_\pi(\frac{1}{t}\frac{d\pi}{1+\pi}) = 1$.

We get $c(K) = \frac{1-p}{p\,|\Delta_K|}\,[K_\infty : F_\infty]$, and we use (i) and (ii) of Remark 3.4 to get $c(K) = -\frac{[K:\mathbf{Q}_p]}{\log \chi(\gamma_K)}$ which concludes the proof. $\square$

3.4. **Tate's formulas.** We will present now a generalization of Tate's computations of the Galois cohomology of C.

3.4.1. *Classical Tate's formulas.* We will often use the following well-known isomorphisms:

$$H^i(\mathscr{G}_K, C(j)) \xleftarrow{\sim} \begin{cases} K & \text{if } j=0 \text{ and } i=0,1,\\ 0 & \text{otherwise.}\end{cases} \tag{3.13}$$

For $i=0$ the above isomorphism is given by the canonical map $K\to H^0(\mathscr{G}_K,C)$; for $i=1$ – by the K-linear map sending 1 to the 1-cocycle $\log\chi$, where χ is the cyclotomic character.

Remark 3.14. The following rescaling of the map from (3.13) will be useful later. Consider the composition

$$\alpha_0:\mathrm{Kos}_\gamma(K)\to \mathrm{Kos}_\gamma(\widehat{K}_\infty)\underset{\sim}{\xleftarrow{\lambda}} C(\Gamma'_K,\widehat{K}'_\infty)\underset{\sim}{\to} C(\Gamma_K,\widehat{K}_\infty)\underset{\sim}{\to} C(\mathscr{G}_K,C),$$

where the last three complexes are complexes of nonhomogeneous condensed cochains[19], and we have

$$\mathrm{Kos}_\gamma(\widehat{K}_\infty)=[\widehat{K}'_\infty\xrightarrow{\gamma_K-1}\widehat{K}'_\infty],\quad \mathrm{Kos}_\gamma(K):=[K\xrightarrow{\gamma_K-1}K]=[K\xrightarrow{0}K].$$

Map λ is given by the identity in degree 0, evaluation on γ_K in degree 1, and 0 in higher degrees. On cohomology level α_0 yields: for $i=0$, the canonical map $K\to H^0(\mathscr{G}_K,C)$; for $i=1$, the K-linear map sending 1 to the 1-cocycle $\frac{\log\chi}{\log\chi(\gamma_K)}$.

3.4.2. *Generalized Tate's formulas.*

Proposition 3.15. *Let $W\in C_K$ be a Banach space, nuclear Fréchet, or a space of compact type equipped with a trivial action of $\mathscr{G}_K$. Then we have isomorphisms:*

$$H^i(\mathscr{G}_K, W(j)\otimes^{\square}_K C) \xleftarrow{\sim} \begin{cases} W & \text{if } j=0 \text{ and } i=0,1,\\ 0 & \text{otherwise.}\end{cases} \tag{3.16}$$

For $j=0$, the maps are defined in the obvious way in the case $i=0$ and as the cup product with the class of $\frac{\log\chi}{\log\chi(\gamma_K)}$ in $H^1(\mathscr{G}_K,C)$ in the case $i=1$.

Proof. Assume first that W is a Banach space. We will use the fact that, if X is profinite and Y is a Banach space, all continuous functions from X to Y^0/p^n (where Y^0 is the unit ball of Y) are locally constant. We can apply this to $X=\mathscr{G}_K\times\cdots\times\mathscr{G}_K$ and use original Tate's arguments to deduce that the inflation map $H^i(\Gamma'_K,\widehat{K}'_\infty\otimes^{\square}_K W)\to H^i(\mathscr{G}_K, C\otimes^{\square}_K W)$ is an isomorphism. Then, we use the fact that $\widehat{K}'_\infty=K\oplus R$ with γ_K-1 invertible with a continuous inverse on R. It follows that $H^i(\Gamma'_K,R\otimes^{\square}_K W)=0$, for all i, and that the canonical map $H^i(\Gamma'_K,W)\to H^i(\Gamma'_K,\widehat{K}'_\infty\otimes^{\square}_K W)$ is an isomorphism. We have proved our proposition in the Banach case.

Assume now that W is a nuclear Fréchet space. Write $W\simeq\lim_n W_n$, for a projective system $\{W_n\}_{n\in\mathbf{N}}$, of Banach spaces with compact transition maps and such that the projections $p_s:\lim_n W_n\to W_s$, $s\in\mathbf{N}$, have dense images (see [30, Ch. 16] for why this is possible). We have

$$\begin{aligned} H^i(\mathscr{G}_K, W(j)\otimes^{\square}_K C) &\simeq H^i(\mathscr{G}_K,(\lim_n W_n(j))\otimes^{\square}_K C)\simeq H^i(\mathscr{G}_K,\lim_n(W_n(j)\otimes^{\square}_K C))\\ &\xrightarrow{\sim} H^i(\mathscr{G}_K,\mathrm{R}\lim_n(W_n(j)\otimes^{\square}_K C))\xrightarrow{\sim}\lim_n H^i(\mathscr{G}_K,W_n(j)\otimes^{\square}_K C).\end{aligned}\tag{3.17}$$

The third isomorphism follows from the fact that the pro-system $\{W_n(j)\otimes^{\square}_K C\}_{n\in\mathbf{N}}$ is Mittag-Leffler (it is a pro-system of C-Banach spaces with projection maps $p_s\otimes\mathrm{Id}$ having dense images). The fourth isomorphism follows from the fact that $\mathrm{R}^1\lim_n H^{i-1}(\mathscr{G}_K,W_n(j)\otimes^{\square}_K C)=0$ since the pro-system $\{H^i(\mathscr{G}_K,W_n(j)\otimes^{\square}_K C)\}_{n\in\mathbf{N}}$ is Mittag-Leffler: by (3.13), this system is or trivial or isomorphic to the pro-system $\{W_n\}_{n\in\mathbf{N}}$, with projection maps p_s having dense images maps.

Having the topological isomorphisms (3.17), by (3.13) we have, in the nontrivial cases, topological isomorphisms

$$\lim_n H^i(\mathscr{G}_K,W_n(j)\otimes^{\square}_K C)\xleftarrow{\sim}\lim_n W_n\xleftarrow{\sim} W,$$

[19]Nonhomogeneous version of (3.2).

as wanted.

Finally, assume that W is of compact type. We write $W \simeq \mathrm{colim}_n W_n$, for an inductive system $\{W_n\}_{n\in\mathbf{N}}$, of Banach spaces with injective, compact transition maps. Then

$$\begin{aligned}(3.18)\quad H^i(\mathscr{G}_K, W(j)\otimes^{\square}_K C) &\simeq H^i(\mathscr{G}_K, (\mathrm{colim}_n W_n(j))\otimes^{\square}_K C) \simeq H^i(\mathscr{G}_K, \mathrm{colim}_n(W_n(j)\otimes^{\square}_K C))\\ &\xleftarrow{\sim} \mathrm{colim}_n H^i(\mathscr{G}_K, W_n(j)\otimes^{\square}_K C).\end{aligned}$$

The third isomorphism follows from the fact that $\mathbf{Z}[\mathscr{G}_K]$ is a compact object in CondAb.

Having the isomorphisms (3.18), by (3.13) we have in the nontrivial cases isomorphisms

$$\mathrm{colim}_n H^i(\mathscr{G}_K, W_n(j)\otimes^{\square}_K C) \xleftarrow{\sim} \mathrm{colim}_n W_n \xrightarrow{\sim} W,$$

as wanted. $\square$

4. Pro-étale cohomology

In this chapter, we study the properties of pro-étale cohomology of smooth dagger curves over K. Moreover we assume that the curve is either proper, or Stein, or a dagger affinoid.

4.1. **Topology on p-adic pro-étale cohomology.**

Let X be a rigid analytic variety over K or C.

4.1.1. *The condensed approach.*

Definition 4.1.
(1) We define the pro-étale site of X as $X_{\mathrm{proét}} := X^{\diamond}_{\mathrm{qproét}}$, where $X^{\diamond}$ is the diamond associated to X and $X^{\diamond}_{\mathrm{qproét}}$ denotes the quasi-pro-étale site of $X^{\diamond}$ [32, Def. 14.1].
(2) For a sheaf $\mathscr{F}$ on $X_{\mathrm{proét}}$ with values in $\mathscr{D}(\mathrm{CondAb})$, the pro-étale cohomology complex

$$\mathrm{R}\Gamma_{\mathrm{proét},\square}(X,\mathscr{F}) \in \mathscr{D}(\mathrm{CondAb}).$$

This is because the category CondAb is closed under all limits and colimits (see Section 2.2.1). The pro-étale cohomology groups $H^i_{\mathrm{proét},\square}(X,\mathscr{F})$, for $i\geq 0$, are objects of CondAb. Similarly, for a sheaf $\mathscr{F}$ with values in $\mathscr{D}(\mathrm{Mod}^{\mathrm{cond}}_{\mathbf{Q}_p})$.

If a sheaf $\mathscr{F}$ in Definition 4.1 has values in $\mathscr{D}(\mathrm{Solid})$ then $\mathrm{R}\Gamma_{\mathrm{proét},\square}(X,\mathscr{F})$ has values in $\mathscr{D}(\mathrm{Solid})$ as well because the category Solid is closed under all limits and colimits (see Section 2.2.2). Similarly, for the category $\mathrm{Mod}^{\mathrm{solid}}_{\mathbf{Q}_p}$.

4.1.2. *Comparison with the classical approach.* Recall that $\mathrm{R}\Gamma(X,\mathbf{Q}_p) := \mathrm{R}\Gamma_{\mathrm{proét}}(X,\mathbf{Q}_p) \in \mathscr{D}(C^{\mathrm{Hcg}}_{\mathbf{Q}_p})$. Locally it is a complex of Banach spaces over $\mathbf{Q}_p$; globally – of Fréchet spaces over $\mathbf{Q}_p$.

Lemma 4.2.
(1) We have a natural quasi-isomorphism in $\mathscr{D}(\mathrm{Mod}^{\mathrm{cond}}_{\mathbf{Q}_p})$

$$\mathrm{CD}(\mathrm{R}\Gamma(X,\mathbf{Q}_p)) \simeq \mathrm{R}\Gamma_{\square}(X,\mathbf{Q}_p).$$

(2) We have a natural isomorphism in $\mathscr{D}(C_{\mathbf{Q}_p})$

$$\mathrm{R}\Gamma_{\square}(X,\mathbf{Q}_p)(*)_{\mathrm{top}} \simeq \mathrm{R}\Gamma(X,\mathbf{Q}_p).$$

(3) We have a natural quasi-isomorphism in $\mathrm{Mod}^{\mathrm{cond}}_{\mathbf{Q}_p}$

$$\mathrm{CD}(\widetilde{H}^i(X,\mathbf{Q}_p)) \simeq H^i_{\square}(X,\mathbf{Q}_p).$$

Proof. Claims (1) and (2) follow from the fact that $\mathrm{R}\Gamma(X,\mathbf{Q}_p)$ is represented locally by Galois cohomology of the fundamental group and we have Lemma 3.3. Claim (3) follows from claim (1) and Section 2.2.3. $\square$

Remark 4.3. By the same arguments, the analog of Lemma 4.2 holds for de Rham cohomology (de Rham complex) as well as for Hyodo-Kato cohomology (see [17, Sec. 4.2] for the definition of the latter).

4.2. **Hochschild-Serre spectral sequence.** We record here the Hochschild-Serre spectral sequence for pro-étale cohomology.

Lemma 4.4. (Bosco, [5, Prop. 4.12]) *Let X be a rigid analytic variety over K. There is a natural Hochschild-Serre spectral sequence*

$$E_2^{a,b}=H^a(\mathscr{G}_K,H^b_{\square}(X_C,\mathbf{Q}_p(j)))\Rightarrow H^{a+b}_{\square}(X,\mathbf{Q}_p(j)). \tag{4.5}$$

Proof. We pass to the world of diamonds. Since $X_C\to X$ is a $\mathscr{G}_K$-torsor, we have isomorphisms

$$(X_C/X)^n\simeq X_C\times\mathscr{G}_K^{n-1},\quad n\geq 1. \tag{4.6}$$

It suffices thus to show that, for any adic space Y over $\mathrm{Spa}(K,\mathscr{O}_K)$ and any profinite set S, we have a natural quasi-isomorphism in $\mathscr{D}(\mathrm{CondAb})$ (take $Y=X_C$ and $S=\mathscr{G}_K^{n-1}$ in the notation from (4.6))

$$\mathrm{R}\Gamma(Y\times S,\mathbf{Q}_p)\simeq \mathrm{R}\underline{\mathrm{Hom}}(\mathbf{Z}_{\square}[S],\mathrm{R}\Gamma(Y,\mathbf{Q}_p)).$$

This is local on Y, hence we may assume that Y is a w-contractible space over $\mathrm{Spa}(K,\mathscr{O}_K)$. Then we have the following natural quasi-isomorphisms

$$\mathrm{R}\Gamma(Y\times S,\mathbf{Q}_p)\simeq\underline{\mathbf{Q}_p(Y\times S)}\simeq\underline{\mathscr{C}(S,\mathbf{Q}_p(Y))}\simeq\underline{\mathrm{Hom}}(\mathbf{Z}[S],\underline{\mathbf{Q}_p(Y)})\simeq\underline{\mathrm{Hom}}(\mathbf{Z}_{\square}[S],\underline{\mathbf{Q}_p(Y)}). \tag{4.7}$$

To see the first quasi-isomorphism note that, for any profinite set T, we have

$$\mathrm{R}\Gamma(Y\times T,\mathbf{Z}/p^n)\simeq\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(Y\times T,\mathbf{Z}/p^n)\simeq\mathbf{Z}/p^n(Y\times T),$$

where the last quasi-isomorphism holds because $Y\times T$ is a strictly totally disconnected perfectoid space (by [32, Lemma 7.19]), and the pro-system $\{\mathbf{Z}/p^n(Y\times T)\}_{n\in\mathbf{N}}$ is Mittag-Leffler. The second quasi-isomorphism in (4.7) follows from the fact that, for any profinite set S',

$$\mathscr{C}(S',\mathscr{C}(S,\mathbf{Q}_p(Y)))\simeq\mathscr{C}(S'\times S,\mathbf{Q}_p(Y)).$$

Now, since $\mathbf{Z}_{\square}[S]$ is an internally projective object in Solid, the right-hand side of (4.7) identifies with

$$\mathrm{R}\underline{\mathrm{Hom}}(\mathbf{Z}_{\square}[S],\underline{\mathbf{Q}_p(Y)})\simeq\mathrm{R}\underline{\mathrm{Hom}}(\mathbf{Z}_{\square}[S],\mathrm{R}\Gamma(Y,\mathbf{Q}_p)),$$

as wanted. $\square$

4.3. **Cohomology of Stein curves.** We will now discuss arithmetic and geometric pro-étale cohomology of smooth Stein curves.

4.3.1. *Geometric cohomology of Stein varieties.* We start with geometric pro-étale cohomology. Let X be a smooth Stein variety over K, geometrically irreducible. By [18, Th. 5.14], $\mathrm{R}\Gamma(X_C,\mathbf{Q}_p(j))\in\mathscr{D}(C_{\mathbf{Q}_p})$ has classical cohomology. Moreover, $H^i(X_C,\mathbf{Q}_p(j))$, for $i\geq 0$, is Fréchet and we have a Galois equivariant strict map of strictly exact sequences of Fréchet spaces

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & \Omega^{i-1}(X_C)/\ker d \arrow[r] \arrow[d, equal] & H^i(X_C,\mathbf{Q}_p(i)) \arrow[r] \arrow[d, "\alpha"] & (H^i_{\mathrm{HK}}(X_C)\widehat{\otimes}_{\breve{F}}\widehat{\mathbf{B}}^+_{\mathrm{st}})^{N=0,\varphi=p^i} \arrow[r] \arrow[d, "\iota_{\mathrm{HK}}\otimes\theta"] & 0\\
0 \arrow[r] & \Omega^{i-1}(X_C)/\ker d \arrow[r, "d"] & \Omega^i(X_C)^{d=0} \arrow[r] & H^i_{\mathrm{dR}}(X_C) \arrow[r] & 0.
\end{tikzcd} \tag{4.8}$$

Warning: These spaces are not nuclear Fréchet over $\mathbf{Q}_p$ (C is not a nuclear Banach space over $\mathbf{Q}_p$ because it is not finite dimensional over $\mathbf{Q}_p$) !

Similarly, in the condensed language, we have the following:

Lemma 4.9. *The cohomology $H^i_{\square}(X_C,\mathbf{Q}_p(j))$, for $i\geq 0$, is Fréchet and we have a Galois equivariant map of exact sequences of Fréchet spaces*

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & \Omega^{i-1}(X_C)/\ker d \arrow[r] \arrow[d, equal] & H^i_{\square}(X_C,\mathbf{Q}_p(i)) \arrow[r] \arrow[d, "\alpha"] & (H^i_{\mathrm{HK},\square}(X_C)\otimes^{\square}_{\breve{F}}\widehat{\mathbf{B}}^+_{\mathrm{st}})^{N=0,\varphi=p^i} \arrow[r] \arrow[d, "\iota_{\mathrm{HK}}\otimes\theta"] & 0\\
0 \arrow[r] & \Omega^{i-1}(X_C)/\ker d \arrow[r, "d"] & \Omega^i(X_C)^{d=0} \arrow[r] & H^i_{\mathrm{dR},\square}(X_C) \arrow[r] & 0.
\end{tikzcd} \tag{4.10}$$

Proof. We apply the functor $\mathrm{CD}(-)$ to the diagram (4.8). Since all the spaces in that diagram are Fréchet, by Lemma 2.18, we obtain a map of exact sequences. By Lemma 4.2 and Remark 4.3, it remains to show that

$$\mathrm{CD}((H^i_{\mathrm{HK}}(X_C)\widehat{\otimes}_{\breve{F}}\widehat{\mathbf{B}}^+_{\mathrm{st}})^{N=0,\varphi=p^i}) \simeq (H^i_{\mathrm{HK},\square}(X_C)\otimes^{\square}_{\breve{F}}\widehat{\mathbf{B}}^+_{\mathrm{st}})^{N=0,\varphi=p^i}.$$

Or, since the functor $\mathrm{CD}(-)$ is left exact, that

$$\mathrm{CD}(H^i_{\mathrm{HK}}(X_C)\widehat{\otimes}_{\breve{F}}\widehat{\mathbf{B}}^+_{\mathrm{st}}) \simeq (H^i_{\mathrm{HK},\square}(X_C)\otimes^{\square}_{\breve{F}}\widehat{\mathbf{B}}^+_{\mathrm{st}}).$$

But, since $H^i_{\mathrm{HK}}(X_C)\simeq \lim_n H^i_{\mathrm{HK}}(X_{n,C})$, where $H^i_{\mathrm{HK}}(X_{n,C})$ are of finite rank over $\breve{F}$, we have

$$\begin{aligned}\mathrm{CD}(H^i_{\mathrm{HK}}(X_C)\widehat{\otimes}_{\breve{F}}\widehat{\mathbf{B}}^+_{\mathrm{st}}) &\simeq \mathrm{CD}(\lim_n H^i_{\mathrm{HK}}(X_{n,C})\widehat{\otimes}_{\breve{F}}\widehat{\mathbf{B}}^+_{\mathrm{st}}) \simeq \lim_n \mathrm{CD}(H^i_{\mathrm{HK}}(X_{n,C})\widehat{\otimes}_{\breve{F}}\widehat{\mathbf{B}}^+_{\mathrm{st}})\\ &\simeq \lim_n(\mathrm{CD}(H^i_{\mathrm{HK}}(X_{n,C}))\otimes^{\square}_{\breve{F}}\widehat{\mathbf{B}}^+_{\mathrm{st}}) \simeq \lim_n(H^i_{\mathrm{HK},\square}(X_{n,C})\otimes^{\square}_{\breve{F}}\widehat{\mathbf{B}}^+_{\mathrm{st}})\\ &\simeq (H^i_{\mathrm{HK},\square}(X_C)\otimes^{\square}_{\breve{F}}\widehat{\mathbf{B}}^+_{\mathrm{st}}),\end{aligned}$$

The second isomorphism follows from the fact that the functor $\mathrm{CD}(-)$ commutes with limits; the third one – from the fact that $H^i_{\mathrm{HK}}(X_{n,C})$ is of finite rank. The last one – from the fact that $\widehat{\mathbf{B}}^+_{\mathrm{st}}$ is Banach and $H^i_{\mathrm{HK},\square}(X_C)$ is a product of spaces of finite rank. □

Notation: From now on we will omit the $\square$ in $\mathrm{R}\Gamma_{\square}(-,-)$ and other cohomologies. This should not cause confusion.

4.3.2. *Arithmetic cohomology of Stein curves.* We pass now to arithmetic pro-étale cohomology. Let X be a smooth Stein curve over K, geometrically irreducible. We will look at its arithmetic pro-étale cohomology complex $\mathrm{R}\Gamma(X,\mathbf{Q}_p(j))\in\mathscr{D}(\mathbf{Q}_{p,\square})$.

Theorem 4.11. *(1) The cohomology of $\mathrm{R}\Gamma(X,\mathbf{Q}_p(j)), j\in\mathbf{Z}$, is nuclear Fréchet.*

(2) Let $\{X_n\}_{n\in\mathbf{N}}$ be a strictly increasing open covering of X by Stein varieties and let $i,j\in\mathbf{Z}$. Then $\mathrm{R}^1\lim_n H^i(X_n,\mathbf{Q}_p(j))=0$. Hence we have an isomorphism

$$H^i(X,\mathbf{Q}_p(j)) \xrightarrow{\sim} \lim_n H^i(X_n,\mathbf{Q}_p(j)).$$

Proof. **Claim (1).** By (4.5), we have the Hochschild-Serre spectral sequence

$$E_2^{a,b}=H^a(\mathscr{G}_K,H^b(X_C,\mathbf{Q}_p(j)))\Rightarrow H^{a+b}(X,\mathbf{Q}_p(j)). \tag{4.12}$$

From diagram (4.10), we know that the only nontrivial cohomology groups of X_C are in degrees 0,1. Hence, from the spectral sequence (4.12), we get that $H^i(X,\mathbf{Q}_p(j))=0$, for $i\geq 4$, and we have the long exact sequence

$$\begin{aligned}0\to&H^0(\mathscr{G}_K,H^0(X_C,\mathbf{Q}_p(j)))\to H^0(X,\mathbf{Q}_p(j))\to H^{-1}(\mathscr{G}_K,H^1(X_C,\mathbf{Q}_p(j)))\\ \to&H^1(\mathscr{G}_K,H^0(X_C,\mathbf{Q}_p(j)))\to H^1(X,\mathbf{Q}_p(j))\to H^0(\mathscr{G}_K,H^1(X_C,\mathbf{Q}_p(j)))\\ \xrightarrow{d_2}&H^2(\mathscr{G}_K,H^0(X_C,\mathbf{Q}_p(j)))\to H^2(X,\mathbf{Q}_p(j))\to H^1(\mathscr{G}_K,H^1(X_C,\mathbf{Q}_p(j)))\\ \to&H^3(\mathscr{G}_K,H^0(X_C,\mathbf{Q}_p(j)))\to H^3(X,\mathbf{Q}_p(j))\to H^2(\mathscr{G}_K,H^1(X_C,\mathbf{Q}_p(j)))\to 0\end{aligned} \tag{4.13}$$

(i) *The groups $H^0(X,\mathbf{Q}_p(j))$ and $H^3(X,\mathbf{Q}_p(j))$.*

Diagram (4.13) yields the isomorphisms

$$\begin{aligned}&H^0(X,\mathbf{Q}_p(j))=\mathbf{Q}_p,\\ &H^3(X,\mathbf{Q}_p(j))\xrightarrow{\sim}H^2(\mathscr{G}_K,H^1(X_C,\mathbf{Q}_p(j))).\end{aligned} \tag{4.14}$$

The top line of diagram (4.10) gives the exact sequence

$$0\to\mathscr{O}(X_C)/C\to H^1(X_C,\mathbf{Q}_p(1))\to \mathrm{HK}^1(X_C,1)\to 0$$

Applying Galois cohomology to it we get the exact sequence (we set $s : j-1$)

(4.15)
$$\begin{aligned} 0 &\to H^0(\mathscr{G}_K,(\mathscr{O}(X_C)/C)(s)) \to H^0(\mathscr{G}_K, H^1(X_C,\mathbf{Q}_p(1))(s)) \to H^0(\mathscr{G}_K,\mathrm{HK}^1(X_C,1)(s)) \\ &\to H^1(\mathscr{G}_K,(\mathscr{O}(X_C)/C)(s)) \to H^1(\mathscr{G}_K, H^1(X_C,\mathbf{Q}_p(1))(s)) \to H^1(\mathscr{G}_K,\mathrm{HK}^1(X_C,1)(s)) \\ &\to H^2(\mathscr{G}_K,(\mathscr{O}(X_C)/C)(s)) \to H^2(\mathscr{G}_K, H^1(X_C,\mathbf{Q}_p(1))(s)) \to H^2(\mathscr{G}_K,\mathrm{HK}^1(X_C,1)(s)) \to 0. \end{aligned}$$

Using it, the isomorphisms (4.14), and the generalized Tate's formulas (3.16), we get the isomorphism

$$H^3(X,\mathbf{Q}_p(j)) \xrightarrow{\sim} H^2(\mathscr{G}_K,\mathrm{HK}^1(X_C,1)(s)). \tag{4.16}$$

(ii) *Key lemma.* Claim (1) of Theorem 4.11 follows from the following fact.

Let $\{X_n\}_{n\in\mathbf{N}}$ be a strictly increasing covering of X by adapted naive interiors of affinoids, i.e., there exists a strictly increasing (Stein) covering $\{\overline{X}_n\}_{n\in\mathbf{N}}$ of X such that X_{n+1} is a naive interior in $\overline{X}_{n+1}$ adapted to X_n.

Remark 4.17. By definition, a *naive interior* of a smooth (dagger) affinoid is a Stein subvariety whose complement is open and quasi-compact. It is easy to see that, for a pair of (dagger) affinoids $X_1 \Subset X_2$ there exists a naive interior $X_2^0 \subset X_2$ such that $X_1 \subset X_2^0 \subset X_2$. We will say that X_2 is *adapted* to X_1.

Lemma 4.18. *The transition maps*

$$f_{i,n}: \quad H^i(X_{n+1},\mathbf{Q}_p(j)) \to H^i(X_n,\mathbf{Q}_p(j)), \quad n \geq 0, \tag{4.19}$$

are compact maps of (nuclear) Fréchet spaces.

Proof. By the computations in (i), which can be applied to each X_n since that variety is Stein, this is clear for $i=0$.

For $i=3$, the isomorphism (4.16) above combined with the fact that $\mathrm{HK}^1(X_{n,C},1)$ is an almost C-representation (because $H^1_{\mathrm{HK}}(X_{n,C})$ is of finite rank over $\breve{F}$ since $H^1_{\mathrm{dR}}(X_{n,C})$ is of finite rank over C) yields that $H^3(X_n,\mathbf{Q}_p(j))$ is a finite rank $\mathbf{Q}_p$-vector space. Hence the maps $f_{3,n}$ are as wanted.

It remains to treat the cases of $i=1,2$. We start with showing that the spaces

$$H^a(\mathscr{G}_K, H^b(X_{n,C},\mathbf{Q}_p(j))), \quad a,b\in\mathbf{Z},$$

appearing the spectral sequence (4.12) are nuclear Fréchet. To see that, we apply Galois cohomology to the top row of diagram (4.10) for X_n and obtain the exact sequence (we set $s := j-b$; $\mathrm{HK}^j(X_{n,C},i) := (H^j_{\mathrm{HK}}(X_{n,C})\otimes^{\square}_{\breve{F}}\widehat{\mathbf{B}}^+_{\mathrm{st}})^{N=0,\varphi=p^i}$)

$$\begin{aligned} &\to H^{a-1}(\mathscr{G}_K,\mathrm{HK}^b(X_{n,C},b)(s)) \xrightarrow{\partial_{a-1}} H^a(\mathscr{G}_K,(\Omega^{b-1}(X_{n,C})/\ker d)(s)) \\ &\to H^a(\mathscr{G}_K, H^b(X_{n,C},\mathbf{Q}_p(j))) \to H^a(\mathscr{G}_K,\mathrm{HK}^b(X_{n,C},b)(s)) \\ &\xrightarrow{\partial_a} H^{a+1}(\mathscr{G}_K,(\Omega^{b-1}(X_{n,C})/\ker d)(s)) \to \end{aligned} \tag{4.20}$$

We claim that the spaces $H^i(\mathscr{G}_K,(\Omega^{b-1}(X_{n,C})/\ker d)(s))$ and $H^i(\mathscr{G}_K,\mathrm{HK}^b(X_{n,C},b)(s))$ are nuclear Fréchet. Indeed, for the first one this follows from the generalized Tate's isomorphism (3.16): if nontrivial

$$H^i(\mathscr{G}_K,(\Omega^{b-1}(X_{n,C})/\ker d)(s)) \simeq \Omega^{b-1}(X_n)/\ker d$$

since $\Omega^{b-1}(X_{n,C})/\ker d \simeq (\Omega^{b-1}(X_n)/\ker d)\otimes^{\square}_K C$; and the fact that $\Omega^{b-1}(X_n)/\ker d$ is a nuclear Fréchet. For the second one, we use the fact that $H^{i-1}(\mathscr{G}_K,\mathrm{HK}^b(X_{n,C},b)(s))$ is a finite rank $\mathbf{Q}_p$-vector space by the isomorphism (3.16) (since $\mathrm{HK}^b(X_{n,C},b)$ is an almost C-representation).

The above computations imply that the maps ∂_{a-1} and ∂_a in (4.20) are between nuclear Fréchet spaces hence $H^a(\mathscr{G}_K, H^b(X_{n,C}, \mathbf{Q}_p(j)))$ is an extension of two nuclear Fréchet spaces. In fact, it is an extension of a finite rank $\mathbf{Q}_p$-vector space by a nuclear Fréchet space hence a nuclear Fréchet space.

We proceed now to prove Lemma 4.18 for $i = 1, 2$. From (4.13), we get a long exact sequence

$$\begin{aligned} (4.21)\qquad & 0 \to H^1(\mathscr{G}_K, \mathbf{Q}_p(j)) \to H^1(X_n, \mathbf{Q}_p(j)) \to H^0(\mathscr{G}_K, H^1(X_{n,C}, \mathbf{Q}_p(j))) \\ & \xrightarrow{d_{2,n}} H^2(\mathscr{G}_K, \mathbf{Q}_p(j)) \to H^2(X_n, \mathbf{Q}_p(j)) \to H^1(\mathscr{G}_K, H^1(X_{n,C}, \mathbf{Q}_p(j))) \to 0 \end{aligned}$$

(a) *Case* $j \neq 1$. In this case, we have $H^2(\mathscr{G}_K, \mathbf{Q}_p(j)) = 0$. Thus, it suffices to show that the spaces $H^i(\mathscr{G}_K, H^1(X_{n,C}, \mathbf{Q}_p(j)))$, for $i = 0, 1$, are of finite rank over $\mathbf{Q}_p$. For that, since the vector spaces $H^i(\mathscr{G}_K, \mathrm{HK}^1(X_{n,C}, 1)(j-1))$, for $i = 0, 1$, are of finite rank over $\mathbf{Q}_p$, it suffices to notice that $H^i(\mathscr{G}_K, (\mathscr{O}(X_{n,C})/C)(j-1)) = 0$, for $i = 0, 1$, by the generalized Tate's formulas (3.16).

(b) *Case* $j = 1$. To start, we claim that the spaces $H^i(X_n, \mathbf{Q}_p(1))$, $i = 1, 2$ are nuclear Fréchet. Indeed, we have shown above that the spaces $H^i(\mathscr{G}_K, H^1(X_{n,C}, \mathbf{Q}_p(j)))$ are Fréchet. We also know that we have exact sequences

$$0 \to V_{0,i,n} \xrightarrow{g_{i,n}} V_{1,i,n} \to H^i(X_n, \mathbf{Q}_p(j)) \to 0,$$

with $V_{0,i,n}, V_{1,i,n}$ solid Fréchet K-vector spaces. Moreover, $H^i(X_n, \mathbf{Q}_p(j))$ is quasi-separated since it is an extension of quasi-separated solid K-vector spaces (see (4.21)). It follows that the map $g_{i,n}$ is quasi-compact and, hence, the induced map $g_{i,n} : V_{0,i,n}(*)_{\mathrm{top}} \to V_{1,i,n}(*)_{\mathrm{top}}$ is a closed embedding. As a result, $V_{1,i,n}(*)_{\mathrm{top}}/V_{0,i,n}(*)_{\mathrm{top}}$ is a (classical) Fréchet space and then this implies that $H^i(X_n, \mathbf{Q}_p(j)) \simeq \mathrm{CD}(V_{1,i,n}(*)_{\mathrm{top}}/V_{0,i,n}(*)_{\mathrm{top}})$ is Fréchet, as wanted. By Lemma 2.5, as an extension of a nuclear Fréchet space by a finite rank vector space, it is nuclear.

We will show below (in fact (c)) that the pro-systems $\{H^i(\mathscr{G}_K, H^1(X_{n,C}, \mathbf{Q}_p(j)))\}_{n\in\mathbf{N}}$ have compact transition maps (we will call such systems *compact*). Then the pro-system $\{\ker d_{2,n}\}_{n\in\mathbf{N}}$ is also compact. And, by Lemma 2.7, the maps $f_{i,n}$, $i = 1, 2$, from our theorem are compact, as wanted.

(c) *The pro-systems* $\{H^i(\mathscr{G}_K, H^1(X_{n,C}, \mathbf{Q}_p(1))(s))\}_{n\in\mathbf{N}}$*, for* $i = 0, 1$*, are compact.* To prove that, note that the vector spaces $H^i(\mathscr{G}_K, \mathrm{HK}^1(X_{n,C}, 1)(s))$, for $i = 0, 1$, are of finite rank over $\mathbf{Q}_p$. Hence, by (4.15), it suffices to show that the pro-systems $\{H^i(\mathscr{G}_K, (\mathscr{O}(X_{n,C})/C)(s))\}_{n\in\mathbf{N}}$ are compact. But this is clear since, by the generalized Tate's formulas (3.16) (note that $\mathscr{O}(X_n)/K$ is a nuclear Fréchet), these pro-systems are or trivial or we have isomorphisms

$$\{H^i(\mathscr{G}_K, (\mathscr{O}(X_{n,C})/C)(s))\}_{n\in\mathbf{N}} \simeq \{\mathscr{O}(X_n)/K\}_{n\in\mathbf{N}}. \quad \square$$

This finishes the proof of claim (1) of the theorem.

Claim (2). It suffices to show that

$$H^i(X, \mathbf{Q}_p(j)) \xrightarrow{\sim} \lim_n H^i(X_n, \mathbf{Q}_p(j)).$$

Or that $\mathrm{R}^1 \lim_n H^i(X_n, \mathbf{Q}_p(j)) = 0$, $i, j \in \mathbf{Z}$. Since the spectral sequence (4.12) degenerates at E_3 it suffices to show that $\mathrm{R}^1 \lim_n E_3^{a,b}(X_n) = 0$. Since cohomological dimension of $\mathscr{G}_K$ is 2, we have

$$E_3^{a,b}(X_n) = \begin{cases} \ker d_2^{0,b}(X_n) & \text{if } a = 0, \\ E_2^{1,b}(X_n) & \text{if } a = 1, \\ \operatorname{coker} d_2^{2,b+1}(X_n) & \text{if } a = 2, \end{cases}$$

where $d_2^{0,b}(X_n) : H^0(\mathscr{G}_K, H^b(X_{n,C}, \mathbf{Q}_p(j))) \to H^2(\mathscr{G}_K, H^{b-1}(X_{n,C}, \mathbf{Q}_p(j)))$ is the only nontrivial differential in the spectral sequence (4.12). Hence, using the computations from claim (1), we get immediately that $\mathrm{R}^1 \lim_n E_3^{1,b}(X_n) = 0$ and, since

$$\mathrm{R}^1 \lim_n H^2(\mathscr{G}_K, H^{b-1}(X_{n,C}, \mathbf{Q}_p(j))) = 0,$$

that $\mathrm{R}^1 \lim_n E_3^{2,b}(X_n) = 0$. It remains to show that

$$\mathrm{R}^1 \lim_n E_3^{0,b}(X_n) = \mathrm{R}^1 \lim_n \ker d_2^{0,b}(X_n) = 0.$$

From the exact sequence (4.20), since $H^0(\mathscr{G}_K, \mathrm{HK}^b(X_{n,C}, b)(s))$ is finite over $\mathbf{Q}_p$ (because $\mathrm{HK}^b(X_{n,C}, b)(s)$ is an almost C-representation), it suffices to show that $\mathrm{R}^1 \lim_n H^0(\mathscr{G}_K, \Omega^{b-1}(X_{n,C})/\ker d) = 0$. But, by the generalized Tate's isomorphism (3.16), $H^0(\mathscr{G}_K, \Omega^{b-1}(X_{n,C})/\ker d) \simeq \Omega^{b-1}(X_n)/\ker d$, so it suffices to show that $\mathrm{R}^1 \lim_n \Omega^{b-1}(X_n) = 0$ but this is known. $\square$

4.4. **Filtration.** Let X be a smooth geometrically irreducible Stein analytic curve over K. Let $i, j \in \mathbf{Z}$. Under certain conditions, there exists an ascending filtration on $H^i(X, \mathbf{Q}_p(j))$:

$$F_{i,j}^2 = H^i(X, \mathbf{Q}_p(j)) \supset F_{i,j}^1 \supset F_{i,j}^0 \supset F_{i,j}^{-1} = 0,$$

such that we have isomorphisms

$$\begin{aligned} F_{i,j}^2/F_{i,j}^1 &\simeq H^{i-1}(\mathscr{G}_K, \mathbf{Q}_p(j-1)),\\ F_{i,j}^1/F_{i,j}^0 &\simeq H^{i-1}(\mathscr{G}_K, \tfrac{\mathscr{O}(X_C)}{C}(j-1)),\\ F_{i,j}^0/F_{i,j}^{-1} &\simeq H^i(\mathscr{G}_K, \mathrm{HK}^1(X_C, 1)(j-1)),\end{aligned}$$

where we set $\mathrm{HK}^1(X_C, 1) = (H^1_{\mathrm{HK}}(X_C) \widehat{\otimes}^{\square}_{\breve{F}} \widehat{\mathbf{B}}^+_{\mathrm{st}})^{N=0,\varphi=p}$. We can visualize this filtration in the following way:

(4.22)

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & F^0_{i,j} := H^i(\mathscr{G}_K, \mathbf{Q}_p(j)) \arrow[r] \arrow[d, equal] & F^1_{i,j} \arrow[r] \arrow[d] & H^{i-1}(\mathscr{G}_K, \frac{\mathscr{O}(X_C)}{C}(j-1)) \arrow[r] \arrow[d] & 0\\
0 \arrow[r] & H^i(\mathscr{G}_K, \mathbf{Q}_p(j)) \arrow[r] & F^2_{i,j} := H^i(X, \mathbf{Q}_p(j)) \arrow[r] \arrow[d] & H^{i-1}(\mathscr{G}_K, H^1(X_C, \mathbf{Q}_p(j))) \arrow[r] \arrow[d] & 0\\
 & & H^{i-1}(\mathscr{G}_K, \mathrm{HK}^1(X_C, 1)(j-1)) \arrow[r, equal] \arrow[d] & H^{i-1}(\mathscr{G}_K, \mathrm{HK}^1(X_C, 1)(j-1)) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

The above diagram is a map of exact sequences. The right column is induced by the syntomic filtration (see diagram (4.10)). The middle row comes from the Hochschild-Serre spectral sequence (4.5) (we note that $\mathbf{Q}_p(j) \simeq H^0(X_C, \mathbf{Q}_p(j))$) and the vanishing of geometric cohomologies obtained from diagram (4.10). We assume that the middle row and the right column are exact. The term $F^1_{i,j}$ is defined as a pullback of the top right square.

4.5. **Arithmetic cohomology of dagger affinoids.** Let X be a smooth geometrically connected dagger affinoid over K. We will now study its arithmetic pro-étale cohomology. They key tool is the (studied above) arithmetic pro-étale cohomology of smooth Stein curves.

Proposition 4.23. *The cohomology of* $\mathrm{R}\Gamma(X, \mathbf{Q}_p(j)), j \in \mathbf{Z}$, *is of compact type.*

Proof. Let $\{X_h\}$ be the dagger presentation of the dagger structure on X. Denote by X_h^0 a naive interior of X_h adapted to $\{X_h\}$. The canonical quasi-isomorphism

$$\mathrm{R}\Gamma(X, \mathbf{Q}_p(j)) \simeq \mathrm{colim}_h\, \mathrm{R}\Gamma(X_h^0, \mathbf{Q}_p(j)), \quad j \in \mathbf{Z}$$

yields an isomorphism

$$H^i(X, \mathbf{Q}_p(j)) \xrightarrow{\sim} \mathrm{colim}_h\, H^i(X_h^0, \mathbf{Q}_p(j)), \quad i, j \in \mathbf{Z}.$$

We note that X_h^0, $h \in \mathbf{N}$, is a smooth Stein variety. By Lemma 4.18, the ind-systems $\{H^i(X_h^0, \mathbf{Q}_p(j))\}_{h \in \mathbf{N}}$, for $i \in \mathbf{N}$, have compact transition maps (between Fréchet spaces). This proves our proposition. $\square$

4.6. **Examples.** In this section we will compute p-adic pro-étale cohomology of open discs, annuli, and their boundaries – the basic building blocks of analytic curves.

4.6.1. *Open disc.* Let D be an open disc over K.

Lemma 4.24. (Geometric cohomology) *Let $j\in\mathbf{Z}$. We have $\mathscr{G}_K$-equivariant isomorphisms*

$$H^i(D_C,\mathbf{Q}_p(j))\simeq\begin{cases}\mathbf{Q}_p(j) & \text{if } i=0,\\ (\mathscr{O}(D_C)/C)(j-1) & \text{if } i=1,\\ 0 & \text{if } i\geq 2.\end{cases}\tag{4.25}$$

Moreover, $d:\mathscr{O}(D_C)/C\to\Omega^1(D_C)$ is an isomorphism.

Proof. The Hyodo-Kato isomorphism $\iota_{\mathrm{HK}}:H^i_{\mathrm{HK}}(D_C)\otimes^{\square}_{\breve{F}}C\simeq H^i_{\mathrm{dR}}(D_C)$ and the fact that $H^i_{\mathrm{dR}}(D_C)=0$, for $i\geq 1$, yield that $H^i_{\mathrm{HK}}(D_C)=0$, for $i\geq 1$. Now, since D is Stein, our lemma follows from diagram (4.10).

The last claim is equivalent to $H^1_{\mathrm{dR}}(D_C)=0$. $\square$

Consider now the Hochschild-Serre spectral sequence (from Lemma 4.4):

$$E_2^{a,b}=H^a(\mathscr{G}_K,H^b(D_C,\mathbf{Q}_p(j)))\Rightarrow H^{a+b}(D,\mathbf{Q}_p(j)).\tag{4.26}$$

By Lemma 4.24, the only nonzero rows are those of degrees $b=0,1$. We get:

Lemma 4.27. (Arithmetic cohomology) *Let $i\geq 0, j\in\mathbf{Z}$. We have exact sequences*

$$0\to H^i(\mathscr{G}_K,H^0(D_C,\mathbf{Q}_p(j)))\to H^i(D,\mathbf{Q}_p(j))\to H^{i-1}(\mathscr{G}_K,H^1(D_C,\mathbf{Q}_p(j)))\to 0,$$
$$0\to H^i(\mathscr{G}_K,\mathbf{Q}_p(j))\to H^i(D,\mathbf{Q}_p(j))\to H^{i-1}(\mathscr{G}_K,(\mathscr{O}(D_C)/C)(j-1))\to 0.$$

Proof. The second exact sequence is a translation of the first, granted formula (4.25). The spectral sequence (4.26) yields the exact sequence

$$\begin{aligned}0\to H^0(\mathscr{G}_K,H^0(D_C,\mathbf{Q}_p(j)))&\to H^0(D,\mathbf{Q}_p(j))\to H^{-1}(\mathscr{G}_K,H^1(D_C,\mathbf{Q}_p(j)))\\ \to H^1(\mathscr{G}_K,H^0(D_C,\mathbf{Q}_p(j)))&\to H^1(D,\mathbf{Q}_p(j))\to H^0(\mathscr{G}_K,H^1(D_C,\mathbf{Q}_p(j)))\\ \xrightarrow{d_2} H^2(\mathscr{G}_K,H^0(D_C,\mathbf{Q}_p(j))&\to H^2(D,\mathbf{Q}_p(j))\to H^1(\mathscr{G}_K,H^1(D_C,\mathbf{Q}_p(j)))\to 0\end{aligned}\tag{4.28}$$

First, we prove that it splits into short exact sequences. If $j\neq 1$, this is trivial as the last terms of the first two lines are 0.

Let us now assume that $j=1$. We need to show that the differential d_2 in (4.28) is 0 or, equivalently, that the canonical map $H^2(\mathscr{G}_K,H^0(D_C,\mathbf{Q}_p(j)))\to H^2(D,\mathbf{Q}_p(j))$ is injective. But this map is induced by the projection $D\to K$ and any rational point in D yields a section (such a point always exists). $\square$

Remark 4.29. We note that the groups $H^i(D_C,\mathbf{Q}_p(j))$ and $H^i(D,\mathbf{Q}_p(j))$ are Fréchet and nuclear Fréchet spaces, by Lemma 4.8 and Theorem 4.11, respectively.

4.6.2. *Open annulus.* Let A be an open annulus over K.

Lemma 4.30. (Geometric cohomology) *Let $j\in\mathbf{Z}$.*

(1) We have $\mathscr{G}_K$-equivariant isomorphisms

$$H^i(A_C,\mathbf{Q}_p(j))\simeq\begin{cases}\mathbf{Q}_p(j) & \text{if } i=0,\\ 0 & \text{if } i\geq 2.\end{cases}\tag{4.31}$$

(2) We have a $\mathscr{G}_K$-equivariant exact sequence

$$0\to(\mathscr{O}(A_C)/C)(j-1)\to H^1(A_C,\mathbf{Q}_p(j))\to\mathbf{Q}_p(j-1)\to 0\tag{4.32}$$

admitting a $\mathscr{G}_K$-equivariant $\mathbf{Q}_p$-linear splitting.

Proof. Since A is Stein we can use the Galois equivariant map of strictly exact sequences (4.8) for $X = A$. From the Hyodo-Kato isomorphism $\iota_{\rm HK} : H^i_{\rm HK}(A_C)\otimes^{\Box}_{\breve{F}} C \simeq H^i_{\rm dR}(A_C)$ and the fact that $H^i_{\rm dR}(A_C) \simeq C$, for $i = 0, 1$, and $H^i_{\rm dR}(A_C) = 0$, for $i \geq 2$, we see that

$$H^0_{\rm HK}(A_C) \simeq \breve{F}, \quad H^1_{\rm HK}(A_C) \simeq \breve{F}, \quad H^i_{\rm HK}(A_C) \simeq 0,\ i \geq 2.$$

The group $H^1_{\rm HK}(A_C)$ is generated by the Hyodo-Kato symbol $c_1^{\rm HK}(z)$, for an arithmetic coordinate z of the annulus. Frobenius acts on $c_1^{\rm HK}(z)$ by multiplication by p and monodromy is trivial. This implies that

$$(H^i_{\rm HK}(A_C)\otimes^{\Box}_{\breve{F}}\widehat{\mathbf{B}}^+_{\rm st})^{N=0,\varphi=p^i} \simeq \begin{cases} \mathbf{Q}_p & \text{if } i = 0, 1, \\ 0 & \text{if } i \geq 2. \end{cases}$$

This yields the isomorphisms in our lemma.

Assume now that $i = 1$. Then the diagram (4.8) becomes

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{O}(A_C)/C \arrow[r] \arrow[d, equal] & H^1(A_C, \mathbf{Q}_p(1)) \arrow[r] \arrow[d, "\alpha"] & \mathbf{Q}_p \arrow[r] \arrow[d, "\mathrm{can}"] & 0 \\ 0 \arrow[r] & \mathscr{O}(A_C)/C \arrow[r] & \Omega^1(A_C) \arrow[r] & C \arrow[r] & 0. \end{tikzcd}$$

The top row yields the exact sequence (4.32). The term $\mathbf{Q}_p$ comes from a Hyodo-Kato term and is generated by the Hyodo-Kato symbol $c_1^{\rm HK}(z)$. The term C comes from de Rham cohomology and is generated by the de Rham symbol $c_1^{\rm dR}(z)$. These symbols are compatible with each other (via the Hyodo-Kato isomorphism $\iota_{\rm HK}$) and are also compatible with the pro-étale symbol $c_1^{\rm pro\acute{e}t}(z)$. Sending $c_1^{\rm HK}(z)$ to $c_1^{\rm pro\acute{e}t}(z)$ yields the wanted splitting of the exact sequence (4.32). $\Box$

Take now the Hochschild-Serre spectral sequence:

$$E_2^{a,b} = H^a(\mathscr{G}_K, H^b(A_C, \mathbf{Q}_p(j))) \Rightarrow H^{a+b}(A, \mathbf{Q}_p(j)). \tag{4.33}$$

By (4.31), the only nonzero rows are those of degrees $b = 0, 1$. We get:

Lemma 4.34. (Arithmetic cohomology) *We have exact sequences*

$$0 \to H^i(\mathscr{G}_K, H^0(A_C, \mathbf{Q}_p(j))) \to H^i(A, \mathbf{Q}_p(j)) \to H^{i-1}(\mathscr{G}_K, H^1(A_C, \mathbf{Q}_p(j))) \to 0,$$
$$0 \to H^{i-1}(\mathscr{G}_K, (\mathscr{O}(A_C)/C)(j-1)) \to H^{i-1}(\mathscr{G}_K, H^1(A_C, \mathbf{Q}_p(j))) \to H^{i-1}(\mathscr{G}_K, \mathbf{Q}_p(j-1)) \to 0.$$

Proof. The second sequence is obtained from the (split) exact sequence (4.32).

For the rest, we argue exactly as in the case of an open disc (see the proof of Lemma 4.27) with the exception of the triviality of the map d_2 in the exact sequence 4.28 when $j = 1$: in this case, a rational point in A does not always exist but it does after taking a base change to a finite extension L of K. Then the map $H^2(\mathscr{G}_L, H^0(A_C, \mathbf{Q}_p(j))) \to H^2(A_L, \mathbf{Q}_p(j))$ is injective and the triviality of d_2 follows. $\Box$

Remark 4.35. We note that the groups $H^i(A_C, \mathbf{Q}_p(j))$ and $H^i(A, \mathbf{Q}_p(j))$ are Fréchet and nuclear Fréchet spaces, by Lemma 4.8 and Theorem 4.11, respectively.

4.6.3. *Ghost circle.* Take now the ghost circle $Y := \partial D$.

Lemma 4.36. (Geometric cohomology) *Let $i \in \mathbf{N}$, $j \in \mathbf{Z}$.*

(1) We have $\mathscr{G}_K$-equivariant isomorphisms

$$H^i(Y_C, \mathbf{Q}_p(j)) \simeq \begin{cases} \mathbf{Q}_p(j) & \text{if } i = 0, \\ 0 & \text{if } i \geq 2. \end{cases}$$

(2) We have a $\mathscr{G}_K$-equivariant (split) exact sequence

$$0 \to (\mathscr{O}(Y_C)/C)(j-1) \to H^1(Y_C, \mathbf{Q}_p(j)) \to \mathbf{Q}_p(j-1) \to 0. \tag{4.37}$$

Proof. By definition, we have

$$\begin{aligned}\mathrm{R}\Gamma(Y_C, \mathbf{Q}_p(j)) &= \mathrm{colim}_{0<\varepsilon<1} \mathrm{R}\Gamma(D_C \setminus D_C(\varepsilon), \mathbf{Q}_p(j)) \\ &= \mathrm{colim}_{0<\varepsilon<1} \mathrm{R}\Gamma(A_C(\varepsilon), \mathbf{Q}_p(j)),\end{aligned}$$

where $D_C(\varepsilon)$ is the closed disc of radius ε (over C) and $A_C(\varepsilon) := D_C \setminus D_C(\varepsilon)$.

Applying Lemma 4.30, we get immediately that $H^i(Y_C, \mathbf{Q}_p(j)) = 0$, for $i \geq 2$, and

$$H^1(Y_C, \mathbf{Q}_p(j)) \xleftarrow{\sim} \mathrm{colim}_{0<\varepsilon<1} H^1(A_C(\varepsilon), \mathbf{Q}_p(j)).$$

From the exact sequence (4.32), we get the exact sequence

$$\begin{aligned}0 &\to \mathrm{colim}_{0<\varepsilon<1}(\mathscr{O}(A_C(\varepsilon)/C)(j-1)) \to \mathrm{colim}_{0<\varepsilon<1} H^1(A_C(\varepsilon), \mathbf{Q}_p(j)) \\ &\to \mathrm{colim}_{0<\varepsilon<1} \mathbf{Q}_p(j-1) \to 0.\end{aligned}$$

Since $\mathrm{colim}_{0<\varepsilon<1}(\mathscr{O}(A_C(\varepsilon)/C)(j-1)) \xrightarrow{\sim} \mathscr{O}(Y_C)/C$, we get that $H^1(Y_C, \mathbf{Q}_p(j))$ fits into the exact sequence (4.37).

From Lemma 4.30, we also get the isomorphism

$$\mathrm{colim}_{0<\varepsilon<1} H^0(A_C(\varepsilon), \mathbf{Q}_p(j)) \xrightarrow{\sim} H^0(Y_C, \mathbf{Q}_p(j)).$$

From it and the exact sequence (4.32), we get the isomorphism

$$\mathbf{Q}_p(j) \xrightarrow{\sim} H^0(Y_C, \mathbf{Q}_p(j)).$$

This finishes the proof of the lemma. □

Lemma 4.38. (Arithmetic cohomology) *Let $i \in \mathbf{N}, j \in \mathbf{Z}$. We have exact sequences*

$$\begin{aligned}&0 \to H^i(\mathscr{G}_K, H^0(Y_C, \mathbf{Q}_p(j))) \to H^i(Y, \mathbf{Q}_p(j)) \to H^{i-1}(\mathscr{G}_K, H^1(Y_C, \mathbf{Q}_p(j))) \to 0, \\ &0 \to H^{i-1}(\mathscr{G}_K, (\mathscr{O}(Y_C)/C)(j-1)) \to H^{i-1}(\mathscr{G}_K, H^1(Y_C, \mathbf{Q}_p(j))) \to H^{i-1}(\mathscr{G}_K, \mathbf{Q}_p(j-1)) \to 0.\end{aligned}$$

Proof. The second exact sequence is obtained from the (split) exact sequence from Lemma 4.36.

For the first exact sequence, we write

$$\begin{aligned}\mathrm{R}\Gamma(Y, \mathbf{Q}_p(j)) &= \mathrm{colim}_{0<\varepsilon_K<1} \mathrm{R}\Gamma(D \setminus D(\varepsilon_K), \mathbf{Q}_p(j)) \\ &= \mathrm{colim}_{0<\varepsilon_K<1} \mathrm{R}\Gamma(A(\varepsilon_K), \mathbf{Q}_p(j)),\end{aligned}$$

where ε_K are chosen so that the annuli $A(\varepsilon_K)$ are defined over K. By Lemma 4.34, this yields the exact sequence

$$\begin{aligned}0 &\to \mathrm{colim}_{0<\varepsilon_K<1} H^i(\mathscr{G}_K, H^0(A(\varepsilon_K)_C, \mathbf{Q}_p(j))) \to H^i(Y, \mathbf{Q}_p(j)) \\ &\to \mathrm{colim}_{0<\varepsilon_K<1} H^{i-1}(\mathscr{G}_K, H^1(A(\varepsilon_K)_C, \mathbf{Q}_p(j))) \to 0.\end{aligned}$$

Hence it suffices to show that

$$\begin{aligned}&\mathrm{colim}_{0<\varepsilon_K<1} H^i(\mathscr{G}_K, H^0(A(\varepsilon_K)_C, \mathbf{Q}_p(j))) \xrightarrow{\sim} H^i(\mathscr{G}_K, H^0(Y_C, \mathbf{Q}_p(j))), \\ &\mathrm{colim}_{0<\varepsilon_K<1} H^{i-1}(\mathscr{G}_K, H^1(A(\varepsilon_K)_C, \mathbf{Q}_p(j))) \xrightarrow{\sim} H^{i-1}(\mathscr{G}_K, H^1(Y_C, \mathbf{Q}_p(j))).\end{aligned} \tag{4.39}$$

But this is clear since $\mathbf{Z}[\mathscr{G}_K]$ is a compact object in CondAb. □

4.6.4. *Boundary of an open annulus.* Let A be an open annulus over K.

Corollary 4.40. (Geometric cohomology) *Let $i \in \mathbf{N}, j \in \mathbf{Z}$. There is a $\mathscr{G}_K$-equivariant canonical isomorphism*

$$H^i(\partial A_C, \mathbf{Q}_p(j)) \xrightarrow{\sim} H^i(Y_C, \mathbf{Q}_p(j))^{\oplus 2}.$$

Hence we have $\mathscr{G}_K$-equivariant isomorphisms

$$H^i(\partial A_C, \mathbf{Q}_p(j)) \simeq \begin{cases} \mathbf{Q}_p(j)^{\oplus 2} & \text{if } i = 0, \\ 0 & \text{if } i \geq 2, \end{cases}$$

and a $\mathscr{G}_K$-equivariant (split) exact sequence:

$$0 \to (\mathscr{O}(Y_C)/C)(j-1)^{\oplus 2} \to H^1(\partial A_C, \mathbf{Q}_p(j)) \to \mathbf{Q}_p(j-1)^{\oplus 2} \to 0$$

Proof. We write $A = \{z \in K \ : \ |a| < |z| < |b|\}$ with $a, b \in K$. Then

$$\partial A \simeq \varinjlim_{|a|<\delta\leq\varepsilon<|b|} A \setminus A(\delta, \varepsilon) = Y_a \sqcup Y_b,$$

where $A(\delta, \varepsilon) := \{z \in K \ : \ \delta \leq |z| \leq \varepsilon\}$ and Y_a, Y_b are the two ghost circles at the boundary of A. Our corollary now follows from Lemma 4.36. $\square$

Corollary 4.41. (Arithmetic cohomology) *We have a canonical isomorphism*

$$H^i(\partial A, \mathbf{Q}_p(j)) \simeq H^i(Y, \mathbf{Q}_p(j))^{\oplus 2}$$

and exact sequences

$$0 \to H^i(\mathscr{G}_K, H^0(\partial A_C, \mathbf{Q}_p(j))) \to H^i(\partial A, \mathbf{Q}_p(j)) \to H^{i-1}(\mathscr{G}_K, H^1(\partial A_C, \mathbf{Q}_p(j))) \to 0,$$
$$0 \to H^{i-1}(\mathscr{G}_K, \tfrac{\mathscr{O}(\partial A_C)}{C^{\oplus 2}}(j-1)) \to H^{i-1}(\mathscr{G}_K, H^1(\partial A, \mathbf{Q}_p(j))) \to H^{i-1}(\mathscr{G}_K, \mathbf{Q}_p^{\oplus 2}(j-1)) \to 0.$$

Proof. Since ∂A is a disjoint union of two ghost circles, this follows immediately from Lemma 4.38. $\square$

Remark 4.42. The arithmetic pro-étale cohomology $H^i(Y, \mathbf{Q}_p(j))$ and $H^i(\partial A, \mathbf{Q}_p(j))$ are direct sums of nuclear Fréchet spaces and spaces of compact type (both over $\mathbf{Q}_p$) (see Proposition 7.25 for an explicit splitting).

5. Pro-étale cohomology with compact support

In this chapter we will study properties of compactly supported pro-étale cohomology of smooth dagger curves.

5.1. **Compactly supported cohomology.** We start with briefly reviewing the definition of pro-étale cohomology with compact support from [1].

5.1.1. *Partially proper varieties.* Let X be a smooth partially proper rigid analytic variety over K, C. We define its p-adic pro-étale cohomology with compact support by:

$$\mathrm{R}\Gamma_c(X, \mathbf{Q}_p(r)) := [\mathrm{R}\Gamma(X, \mathbf{Q}_p(r)) \to \mathrm{R}\Gamma(\partial X, \mathbf{Q}_p(r))] \in \mathscr{D}(\mathbf{Q}_p), \quad r \geq 0, \tag{5.1}$$

with

$$\mathrm{R}\Gamma(\partial X, \mathbf{Q}_p(r)) := \mathrm{colim}_Z\, \mathrm{R}\Gamma(X \setminus Z, \mathbf{Q}_p(r)) \in \mathscr{D}(\mathbf{Q}_p),$$

where the colimit is taken over admissible quasi-compact opens $Z \subset X$. From the definition, we get a distinguished triangle

$$\mathrm{R}\Gamma_c(X, \mathbf{Q}_p(r)) \to \mathrm{R}\Gamma(X, \mathbf{Q}_p(r)) \to \mathrm{R}\Gamma(\partial X, \mathbf{Q}_p(r)).$$

By [1, Sec. 2.1], $\mathrm{R}\Gamma_c(X, \mathbf{Q}_p(r))$ is a cosheaf for the analytic topology on X.

If X is a proper variety then the definition (5.1) yields that we have the canonical isomorphism

$$\mathrm{R}\Gamma_c(X, \mathbf{Q}_p(r)) \xrightarrow{\sim} \mathrm{R}\Gamma(X, \mathbf{Q}_p(r))$$

and the cohomology groups of $\mathrm{R}\Gamma(X, \mathbf{Q}_p(j))$ are classical: they are finite dimensional vector spaces over $\mathbf{Q}_p$ equipped with their canonical Hausdorff topology. By Lemma 4.2 so is the cohomology of the complex $\mathrm{R}\Gamma_\square(X, \mathbf{Q}_p(j))$ (which can be identified with the cohomology of $\mathrm{R}\Gamma(X, \mathbf{Q}_p(j))$ via the functor CD).

Remark 5.2. The de Rham and Hyodo-Kato cohomologies with compact support $\mathrm{R}\Gamma_{\mathrm{dR},c}(X)$ and $\mathrm{R}\Gamma_{\mathrm{HK},c}(X)$ can be defined in an analogous way (see [7]).

5.1.2. *Dagger affinoids.* Let X be a smooth dagger affinoid over K,C with a presentation $\{X_h\}$. We set

$$\mathrm{R}\Gamma_c(X,\mathbf{Q}_p(r)) := \mathrm{R}\lim_h \mathrm{R}\Gamma_c(X_h^0,\mathbf{Q}_p(r)) \in \mathscr{D}(\mathbf{Q}_{p,\square}),$$

where X_h^0 denotes a naive interior[20] of X_h adapted to the presentation $\{X_h\}$. This definition is independent of the interiors chosen. Alternatively, we can set

$$\mathrm{R}\Gamma_c(X,\mathbf{Q}_p(r)) := \mathrm{R}\Gamma_{\widehat{X}}(X_h,\mathbf{Q}_p(r)) \in \mathscr{D}(\mathbf{Q}_{p,\square}).$$

This is independent of h.

5.2. **Hochschild-Serre spectral sequence.** We record the Hochschild-Serre spectral sequence for pro-étale cohomology with compact support.

Lemma 5.3. *Let X be a smooth partially proper variety over K. There is a spectral sequence*

$$E_2^{a,b} = H^a(\mathscr{G}_K, H^b_c(X_C,\mathbf{Q}_p(j))) \Rightarrow H^{a+b}_c(X,\mathbf{Q}_p(j)). \tag{5.4}$$

Proof. By definition (see (5.1)), we have

$$\begin{aligned}\mathrm{R}\Gamma_c(X,\mathbf{Q}_p) &= [\mathrm{R}\Gamma(X,\mathbf{Q}_p)\to \mathrm{R}\Gamma(\partial X,\mathbf{Q}_p)],\\ \mathrm{R}\Gamma(\partial X,\mathbf{Q}_p) &:= \mathrm{colim}_Z\,\mathrm{R}\Gamma(X\setminus Z,\mathbf{Q}_p),\end{aligned}$$

where the colimit is taken over admissible quasi-compact opens Z in X. This yields natural quasi-isomorphisms

$$\begin{aligned}\mathrm{R}\Gamma_c(X,\mathbf{Q}_p) &= [\mathrm{R}\Gamma(X,\mathbf{Q}_p)\to \mathrm{R}\Gamma(\partial X,\mathbf{Q}_p)]\\ &\simeq [\mathrm{R}\Gamma(\mathscr{G}_K,\mathrm{R}\Gamma(X_C,\mathbf{Q}_p))\to \mathrm{R}\Gamma(\mathscr{G}_K,\mathrm{R}\Gamma(\partial X_C,\mathbf{Q}_p))]\\ &\simeq \mathrm{R}\Gamma(\mathscr{G}_K,[\mathrm{R}\Gamma(X_C,\mathbf{Q}_p)\to \mathrm{R}\Gamma(\partial X_C,\mathbf{Q}_p)])\\ &= \mathrm{R}\Gamma(\mathscr{G}_K,\mathrm{R}\Gamma_c(X_C,\mathbf{Q}_p)).\end{aligned}$$

The second quasi-isomorphism needs a justification. The quasi-isomorphism involving X follows from Lemma 4.4. For ∂X we have quasi-isomorphisms

$$\begin{aligned}\mathrm{R}\Gamma(\partial X,\mathbf{Q}_p) &= \mathrm{colim}_Z\,\mathrm{R}\Gamma(X\setminus Z,\mathbf{Q}_p)\simeq \mathrm{colim}_Z\,\mathrm{R}\Gamma(\mathscr{G}_K,\mathrm{R}\Gamma((X\setminus Z)_C,\mathbf{Q}_p))\\ &\simeq \mathrm{R}\Gamma(\mathscr{G}_K,\mathrm{colim}_Z\,\mathrm{R}\Gamma((X\setminus Z)_C,\mathbf{Q}_p))\simeq \mathrm{R}\Gamma(\mathscr{G}_K,\mathrm{R}\Gamma(\partial X_C,\mathbf{Q}_p)).\end{aligned}$$

The second quasi-isomorphism follows from Lemma 4.4; the third one from the fact that $\mathbf{Z}[\mathscr{G}_K^i]$ is a compact object in CondAb. $\square$

5.3. **Compactly supported cohomology of Stein curves.** We turn now to the study of compactly supported pro-étale cohomology of smooth Stein curves.

5.3.1. *Geometric compactly supported cohomology.* We will briefly review here computations of geometric compactly supported cohomology from [1, Sec. 8.2].

Let X be a geometrically connected smooth Stein curve over K. Its geometric compactly supported pro-étale cohomology $\mathrm{R}\Gamma_c(X_C,\mathbf{Q}_p(j))\in\mathscr{D}(\mathbf{Q}_{p,\square})$ is studied in loc. cit. by using a Galois equivariant exact sequence ($i\geq 0$)

$$H^{i-1}\mathrm{HK}_c(X_C,i)\longrightarrow H^{i-1}\mathrm{DR}_c(X_C,i)\longrightarrow H^i_c(X_C,\mathbf{Q}_p(i))\longrightarrow H^i\mathrm{HK}_c(X_C,i)\longrightarrow H^i\mathrm{DR}_c(X_C,i),$$

where we set

$$\begin{aligned}\mathrm{HK}_c(X_C,i)\mathrm{DR}_c(X_C,i) &:= (\mathrm{R}\Gamma_{\mathrm{dR},c}(X)\otimes^{\mathrm{L}\square}_K\mathbf{B}^+_{\mathrm{dR}})/F^i\\ &\simeq \big(H^1_c(X,\mathscr{O}_X)\otimes^{\square}_K(\mathbf{B}^+_{\mathrm{dR}}/F^i)\to H^1_c(X,\Omega^1_X)\otimes^{\square}_K(\mathbf{B}^+_{\mathrm{dR}}/F^{i-1})\big)[-1].\end{aligned}$$

This sequence is obtained from a comparison theorem between pro-étale cohomology and syntomic cohomology. It yields the following facts:

[20] See Remark 4.17 for a discussion.

Lemma 5.5. *We have*

(1) *vanishings:* $H^i_c(X_C,\mathbf{Q}_p)=0$ *for* $i\neq 1,2$.

(2) *an isomorphism:*

$$H^1_c(X_C,\mathbf{Q}_p(1))\xrightarrow{\sim}(H^1_{\mathrm{HK},c}(X_C)\otimes^{\square}_{\breve{F}}\widehat{\mathbf{B}}^+_{\mathrm{st}})^{N=0,\varphi=1}. \tag{5.6}$$

(3) *an exact sequence:*

$$H^1\mathrm{HK}_c(X_C,2)\longrightarrow H^1\mathrm{DR}_c(X_C,2)\longrightarrow H^2_c(X_C,\mathbf{Q}_p(2))\longrightarrow\mathbf{Q}_p(1)\longrightarrow 0. \tag{5.7}$$

The above map $\mathrm{Tr}_{X_C}:H^2_c(X_C,\mathbf{Q}_p(2))\to\mathbf{Q}_p(1)$ is *the geometric trace map.*

5.3.2. *Arithmetic compactly supported cohomology.* Let X be a geometrically connected smooth Stein curve over K. We will now look at its arithmetic compactly supported pro-étale cohomology complex $\mathrm{R}\Gamma_c(X,\mathbf{Q}_p(j))\in\mathscr{D}(\mathbf{Q}_{p,\square})$.

Theorem 5.8. *The cohomology of* $\mathrm{R}\Gamma_c(X,\mathbf{Q}_p(j))$ *is of compact type.*

Proof. The only nontrivial geometric cohomology groups are in degrees 1,2 hence, from the spectral sequence (5.4), we get that $H^0_c(X,\mathbf{Q}_p(j))=0$ and we have the long exact sequence

$$\begin{aligned}0\to&H^0(\mathscr{G}_K,H^1_c(X_C,\mathbf{Q}_p(j)))\to H^1_c(X,\mathbf{Q}_p(j))\to H^{-1}(\mathscr{G}_K,H^2_c(X_C,\mathbf{Q}_p(j)))\\ \to&H^1(\mathscr{G}_K,H^1_c(X_C,\mathbf{Q}_p(j)))\to H^2_c(X,\mathbf{Q}_p(j))\to H^0(\mathscr{G}_K,H^2_c(X_C,\mathbf{Q}_p(j)))\\ \xrightarrow{d_2}&H^2(\mathscr{G}_K,H^1_c(X_C,\mathbf{Q}_p(j)))\to H^3_c(X,\mathbf{Q}_p(j))\to H^1(\mathscr{G}_K,H^2_c(X_C,\mathbf{Q}_p(j)))\\ \to&H^3(\mathscr{G}_K,H^1_c(X_C,\mathbf{Q}_p(j)))\to H^4_c(X,\mathbf{Q}_p(j))\to H^2(\mathscr{G}_K,H^2_c(X_C,\mathbf{Q}_p(j)))\to 0\end{aligned}$$

It follows that

$$\begin{aligned}&H^0_c(X,\mathbf{Q}_p(j)))=0,\\ &H^1_c(X,\mathbf{Q}_p(j))\xrightarrow{\sim}H^0(\mathscr{G}_K,H^1_c(X_C,\mathbf{Q}_p(j))),\\ &H^4_c(X,\mathbf{Q}_p(j))\xrightarrow{\sim}H^2(\mathscr{G}_K,H^2_c(X_C,\mathbf{Q}_p(j))).\end{aligned}$$

Hence, by (5.6),

$$H^1_c(X,\mathbf{Q}_p(j))\simeq H^0(\mathscr{G}_K,\mathrm{HK}^1_c(X_C,1)(j-1)),$$

where we set $\mathrm{HK}^1_c(X_C,1):=H^1\mathrm{HK}_c(X_C,1)$. It follows that $H^1_c(X,\mathbf{Q}_p(j))$ is a colimit of finite rank $\mathbf{Q}_p$-vector spaces hence of compact type. By [1, Sec. 8.3], we have that

$$H^4_c(X,\mathbf{Q}_p(j))\simeq\begin{cases}\mathbf{Q}_p & \text{if } j=2,\\ 0 & \text{otherwise.}\end{cases}$$

It remains to treat $H^i_c(X,\mathbf{Q}_p(j))$, for $i=2,3$. We have a long exact sequence

$$\begin{aligned}0\to&H^1(\mathscr{G}_K,H^1_c(X_C,\mathbf{Q}_p(j)))\to H^2_c(X,\mathbf{Q}_p(j))\to H^0(\mathscr{G}_K,H^2_c(X_C,\mathbf{Q}_p(j)))\\ \xrightarrow{d_2}&H^2(\mathscr{G}_K,H^1_c(X_C,\mathbf{Q}_p(j)))\to H^3_c(X,\mathbf{Q}_p(j))\to H^1(\mathscr{G}_K,H^2_c(X_C,\mathbf{Q}_p(j)))\to 0\end{aligned} \tag{5.9}$$

(•) *Case* $j\neq 2$. We claim that in this case $H^2(\mathscr{G}_K,H^1_c(X_C,\mathbf{Q}_p(j)))=0$. Indeed, we have

$$H^2(\mathscr{G}_K,H^1_c(X_C,\mathbf{Q}_p(j)))\simeq H^2(\mathscr{G}_K,\mathrm{HK}^1_c(X_C,1)(j-1)).$$

Now the slopes of Frobenius on $H^1_{\mathrm{HK},c}(X_C)$ are between 0 and 1, which implies that, in the case this group is of finite rank, $\mathrm{HK}^1_c(X_C,1)$ is an extension of an unramified finite dimensional $\mathbf{Q}_p$-representation V of $\mathscr{G}_K$ by the C-points W of a connected BC. Since we have $H^2(\mathscr{G}_K,W(j-1))=0$, for any j, and $H^2(\mathscr{G}_K,V(j-1))=0$, for $j\neq 2$, this implies $H^2(\mathscr{G}_K,\mathrm{HK}^1_c(X_C,1)(j-1))=0$, as wanted. The general case is obtained by writing $H^1_{\mathrm{HK},c}(X_C)$ as a colimit of groups of finite rank.

(i) *First sequence.* Let us now look at the first exact sequence from (5.9).

$$0\to H^1(\mathscr{G}_K,H^1_c(X_C,\mathbf{Q}_p(j)))\to H^2_c(X,\mathbf{Q}_p(j))\to H^0(\mathscr{G}_K,H^2_c(X_C,\mathbf{Q}_p(j)))\to 0 \tag{5.10}$$

Let $\{X_n\}_{n\in\mathbf{N}}$ be a strictly increasing open covering of X by adapted naive interiors of dagger affinoids. We have an isomorphism

$$\mathrm{colim}_n\, H^i_c(X_n,\mathbf{Q}_p(j)) \xrightarrow{\sim} H^i_c(X,\mathbf{Q}_p(j)),\quad i,j\in\mathbf{Z}.$$

We have analogs of sequence (5.10) for X_n's.

It suffices to prove the following result.

Lemma 5.11. *The transition maps $f_{2,n}: H^2_c(X_n,\mathbf{Q}_p(j))\to H^2_c(X_{n+1},\mathbf{Q}_p(j))$ are compact maps between spaces of compact type.*

Proof. Consider the canonical transition map

$$f_{1,n}: H^1(\mathscr{G}_K, H^1_c(X_{n,C},\mathbf{Q}_p(j)))\to H^1(\mathscr{G}_K, H^1_c(X_{n+1,C},\mathbf{Q}_p(j))).$$

We have

$$H^1(\mathscr{G}_K, H^1_c(X_{n,C},\mathbf{Q}_p(j)))\simeq H^1(\mathscr{G}_K, \mathrm{HK}^1_c(X_{n,C},1)(j-1)).$$

Hence it is of finite rank over $\mathbf{Q}_p$. This implies, by Lemma 2.5, Lemma 2.7, and the exact sequence (5.10), that it suffices to show that the canonical transition map

$$f_{3,n}: H^0(\mathscr{G}_K, H^2_c(X_{n,C},\mathbf{Q}_p(j)))\to H^0(\mathscr{G}_K, H^2_c(X_{n+1,C},\mathbf{Q}_p(j)))$$

is a compact map of spaces of compact type.

By (5.7), we have the exact sequence, for $s=n,n+1$,

$$0\to H^0(\mathscr{G}_K,(\mathrm{coker}\, g(X_s))(j-2))\to H^0(\mathscr{G}_K, H^2_c(X_{s,C},\mathbf{Q}_p(2))(j-2))\to H^0(\mathscr{G}_K,\mathbf{Q}_p(j-1))\to,$$

where $g(X_s): H^1\mathrm{HK}_c(X_{s,C},2)\to H^1\mathrm{DR}_c(X_{s,C},2)$ is the canonical map. But

$$\mathrm{coker}\, g(X_s)\simeq \mathrm{coker}((H^1_{\mathrm{HK},c}(X_{s,C})\otimes^{\square}_{\breve{F}} t\widehat{\mathbf{B}}^+_{\mathrm{st}})^{N=0,\varphi=p^2}\to H^1_{\mathrm{dR},c}(X_{s,C})\otimes^{\square}_K C(1)\hookrightarrow H^1_c(X_s,\mathscr{O}_{X_s})\otimes^{\square}_K C(1)).$$

Now, using generalized Tate's formulas (see (3.16)), we easily see that $H^0(\mathscr{G}_K,(\mathrm{coker}\, g(X_s))(j-2))$ is of finite rank over $\mathbf{Q}_p$ unless $j=1$. Hence, for $j\neq 1$, the map $f_{3,n}$ is a map between finite rank vector spaces over $\mathbf{Q}_p$; hence it is compact.

Assume now that $j=1$. It suffices to show that the transition maps between the spaces $H^0(\mathscr{G}_K,(\mathrm{coker}\, g(X_n))(-1))$ are compact and the spaces themselves are of compact type. From the definition of the map $g(X_n)$, we get an exact sequence

$$0\to A_n\to H^1_c(X_n,\mathscr{O}_{X_n})\otimes^{\square}_K C\to(\mathrm{coker}\, g(X_n))(-1)\to 0, \tag{5.12}$$

where A_n is an almost C-representation. Applying Galois cohomology to this sequence, we get the exact sequence

$$0\to H^0(\mathscr{G}_K,A_n)\to H^1_c(X_n,\mathscr{O}_{X_n})\to H^0(\mathscr{G}_K,(\mathrm{coker}\, g(X_n))(-1))\to H^1(\mathscr{G}_K,A_n).$$

Since, by generalized Tate's theorem (see (3.16)), $H^0(\mathscr{G}_K,A_n)$ and $H^1(\mathscr{G}_K,A_n)$ are of finite rank over $\mathbf{Q}_p$, it suffices to show that the canonical map $f: H^1_c(X_n,\mathscr{O}_{X_n})\to H^1_c(X_{n+1},\mathscr{O}_{X_{n+1}})$ is compact and the cokernel of the map $g: H^0(\mathscr{G}_K,A_n)\to H^1_c(X_n,\mathscr{O}_{X_n})$ is of compact type. For that, choose rigid analytic affinoids Y_1, Y_1 such that

$$X_n\subset Y_1\Subset Y_2\subset X_{n+1}.$$

Then we have maps

$$\Omega(X_{n+1})\to\Omega(Y_2)\xrightarrow{\tilde{f}}\Omega(Y_1)\to\Omega(X_n). \tag{5.13}$$

The spaces $\Omega(Y_i)$, $i=1,2$, are Banach and the map $\tilde{f}$ is compact. Taking the strong duals of the terms of (5.13) and using Serre's duality for the end terms we get maps

$$H^1_c(X_n,\mathscr{O}_{X_n})\to\Omega(Y_1)^*\xrightarrow{\tilde{f}^*}\Omega(Y_2)^*\to H^1_c(X_{n+1},\mathscr{O}_{X_{n+1}})$$

factorizing the map f. By [30, Lemma 16.4], the map $\tilde{f}^*$ is compact; by [30, Remark 16.7], so is the map f, as wanted.

The statement about the cokernel of the map g follows from the fact that the space $H^1_c(X_n,\mathscr{O}_{X_n})$ is of compact type and the image of g is finite dimensional. $\square$

(ii) *Second sequence.* We pass now to the second exact sequence from (5.9) which became the isomorphism

$$H^3_c(X,\mathbf{Q}_p(j))\xrightarrow{\sim} H^1(\mathscr{G}_K,H^2_c(X_C,\mathbf{Q}_p(j))).$$

We cover X with a strictly increasing system $\{X_n\}_{n\in\mathbf{N}}$ of adapted naive interiors of dagger affinoids. It suffices to show that the transition maps

$$f_n:\quad H^1(\mathscr{G}_K,H^2_c(X_{n,C},\mathbf{Q}_p(j)))\to H^1(\mathscr{G}_K,H^2_c(X_{n+1,C},\mathbf{Q}_p(j)))$$

are compact maps between spaces of compact type.

By (5.7), we have the exact sequence

$$H^0(\mathscr{G}_K,\mathbf{Q}_p(j-1))\to H^1(\mathscr{G}_K,(\operatorname{coker} g(X_s))(j-2))\to H^1(\mathscr{G}_K,H^2_c(X_{s,C},\mathbf{Q}_p(2))(j-2))\to H^1(\mathscr{G}_K,\mathbf{Q}_p(j-1)).$$

Using the generalized Tate's formulas (see (3.16)), we easily see that $H^1(\mathscr{G}_K,(\operatorname{coker} g(X_s))(j-2))$ is of finite rank over $\mathbf{Q}_p$ unless $j=1$. Hence, for $j\neq 1$, the map f_n is compact as a map between finite rank vector spaces over $\mathbf{Q}_p$.

Assume now that $j=1$. It suffices to show that the transition maps between the spaces $H^1(\mathscr{G}_K,(\operatorname{coker} g(X_n))(-1))$ are maps of compact type between spaces of compact type. By generalized Tate's theorem (see (3.16)), from the exact sequence (5.12), we get the exact sequence

$$\to H^1(\mathscr{G}_K,A_n)\to H^1_c(X_n,\mathscr{O}_{X_n})\to H^1(\mathscr{G}_K,(\operatorname{coker} g(X_n))(-1))\to H^2(\mathscr{G}_K,A_n)\to .$$

Since $H^1(\mathscr{G}_K,A_n)$ and $H^2(\mathscr{G}_K,A_n)$ are of finite rank over $\mathbf{Q}_p$, it suffices to show that the canonical map $H^1_c(X_n,\mathscr{O}_{X_n})\to H^1_c(X_{n+1},\mathscr{O}_{X_{n+1}})$ is a compact map of spaces of compact type but this we have done above.

($\bullet$) *Case $j=2$.* We cover X with a strictly increasing system $\{X_n\}_{n\in\mathbf{N}}$ of adapted naive interiors of dagger affinoids. It suffices to show that all the terms in the exact sequence (5.9) on level X_n are of finite rank over $\mathbf{Q}_p$ (this is true more generally for $j\neq 1$). But, by (5.7), since $H^0(\mathscr{G}_K,\mathbf{Q}_p(1))=0$, we have

$$H^0(\mathscr{G}_K,\operatorname{coker} g(X_n))\xrightarrow{\sim} H^0(\mathscr{G}_K,H^2_c(X_{n,C},\mathbf{Q}_p(2))).$$

And we have shown above that this is of finite rank over $\mathbf{Q}_p$. $\square$

5.4. **Arithmetic cohomology of dagger affinoids.** Let X be a smooth geometrically connected dagger affinoid over K. We will now study its arithmetic pro-étale cohomology. They key tool is the (studied above) arithmetic pro-étale cohomology of smooth Stein curves.

Proposition 5.14. *The cohomology of $\mathrm{R}\Gamma_c(X,\mathbf{Q}_p(j)), j\in\mathbf{Z}$, is nuclear Fréchet. Moreover, if $\{X_h\}$ is the dagger presentation of the dagger structure on X, then we have an isomorphism*

$$H^i_c(X,\mathbf{Q}_p(j))\xrightarrow{\sim}\lim_h H^i_c(X^0_h,\mathbf{Q}_p(j)),\quad i,j\in\mathbf{Z},$$

where X^0_h is a naive interior of X_h adapted to $\{X_h\}$.

Proof. Let $\{X_h\}$ be the dagger presentation of the dagger structure on X. Denote by X^0_h a naive interior of X_h adapted to $\{X_h\}$. Note that X^0_h is a smooth and Stein rigid analytic variety over K. From Section 5.3.2, we have

$$\begin{aligned}
&H^0_c(X^0_h,\mathbf{Q}_p(j)))=0,\\
&H^1_c(X^0_h,\mathbf{Q}_p(j))\xrightarrow{\sim} H^0(\mathscr{G}_K,H^1_c(X^0_{h,C},\mathbf{Q}_p(j)))\simeq H^0(\mathscr{G}_K,\mathrm{HK}^1_c(X^0_{h,C},1)(j-1)),\\
&H^4_c(X^0_h,\mathbf{Q}_p(j))\xrightarrow{\sim} H^2(\mathscr{G}_K,H^2_c(X^0_{h,C},\mathbf{Q}_p(j)))\simeq\begin{cases}\mathbf{Q}_p & \text{if } j=2,\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

These are finite rank $\mathbf{Q}_p$-vector spaces. Moreover, by Lemma 5.11, the transition maps

$$H^i_c(X^0_h, \mathbf{Q}_p(j)) \to H^i_c(X^0_{h+1}, \mathbf{Q}_p(j)), \quad i = 2, 3,$$

are compact maps between spaces of compact type.

By [30, discussion after Prop. 16.5], since the pro-system $\{H^i_c(X^0_h, \mathbf{Q}_p(j))\}_{h\in\mathbf{N}}$, $i \geq 0$, is compact, it is equivalent to a pro-system of Banach spaces with dense transition maps. Hence it is Mittag-Leffler by Section 2.2.4 and we have $\mathrm{R}^1 \lim_h H^i_c(X^0_h, \mathbf{Q}_p(j)) = 0$, $i \geq 0$. It follows that

$$H^i_c(X, \mathbf{Q}_p(j)) \xrightarrow{\sim} \lim_h H^i_c(X^0_h, \mathbf{Q}_p(j))$$

and the cohomology of $\mathrm{R}\Gamma_c(X, \mathbf{Q}_p(j)), j \in \mathbf{Z}$, is nuclear Fréchet, as wanted. □

5.5. **Examples.** We will now determine the compactly supported p-adic pro-étale cohomology groups of an open disc and an open annulus.

5.5.1. *Open disc.* We start with the cohomology $H^i_c(D, \mathbf{Q}_p(j))$ of an open disc D over K. It immediately follows from Lemma 4.24 and Lemma 4.36, and the definition of compactly supported cohomology that we have:

Lemma 5.15. (Geometric cohomology) *Let $i \in \mathbf{N}, j \in \mathbf{Z}$. Then:*

(1) *The canonical maps $H^i(D_C, \mathbf{Q}_p(j)) \to H^i(Y_C, \mathbf{Q}_p(j))$ are injective; hence we have a $\mathscr{G}_K$-equivariant isomorphism*

$$H^i_c(D_C, \mathbf{Q}_p(j)) \xleftarrow{\sim} H^{i-1}(Y_C, \mathbf{Q}_p(j))/H^{i-1}(D_C, \mathbf{Q}_p(j)).$$

(2) *The groups $H^i_c(D_C, \mathbf{Q}_p(j))$ are 0 unless $i = 2$, in which case there is a $\mathscr{G}_K$-equivariant (split) exact sequence:*

$$0 \to (\mathscr{O}(Y_C)/\mathscr{O}(D_C))(j-1) \to H^2_c(D_C, \mathbf{Q}_p(j)) \to \mathbf{Q}_p(j-1) \to 0.$$

Now we pass to the arithmetic cohomology. The canonical maps

$$H^i(D, \mathbf{Q}_p(j)) \to H^i(Y, \mathbf{Q}_p(j)) \tag{5.16}$$

are injective; this follows from the descriptions of these groups in Lemma 4.27 and Lemma 4.38. Hence we have an isomorphism

$$H^i_c(D, \mathbf{Q}_p(j)) \xleftarrow{\sim} H^{i-1}(Y, \mathbf{Q}_p(j))/H^{i-1}(D, \mathbf{Q}_p(j)). \tag{5.17}$$

Lemma 5.18. (Arithmetic Cohomology) *Let $i \in \mathbf{N}, j \in \mathbf{Z}$. There is an exact sequence:*

$$0 \to H^{i-2}(\mathscr{G}_K, \tfrac{\mathscr{O}(Y_C)}{\mathscr{O}(D_C)}(j-1)) \to H^i_c(D, \mathbf{Q}_p(j)) \to H^{i-2}(\mathscr{G}_K, \mathbf{Q}_p(j-1)) \to 0$$

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & H^{i-1}(\mathscr{G}_K, \mathbf{Q}_p(j)) \arrow[r] \arrow[d, "\wr"] & H^{i-1}(D, \mathbf{Q}_p(j)) \arrow[r] \arrow[d] & H^{i-2}(\mathscr{G}_K, H^1(D_C, \mathbf{Q}_p(j))) \arrow[r] \arrow[d, "f_1"] & 0 \\
0 \arrow[r] & H^{i-1}(\mathscr{G}_K, \mathbf{Q}_p(j)) \arrow[r] & H^{i-1}(Y, \mathbf{Q}_p(j)) \arrow[r] \arrow[d] & H^{i-2}(\mathscr{G}_K, H^1(Y_C, \mathbf{Q}_p(j))) \arrow[r] \arrow[d, "f_2"] & 0 \\
 & & H^i_c(D, \mathbf{Q}_p(j)) \arrow[r] \arrow[d] & H^{i-2}(\mathscr{G}_K, H^2_c(D_C, \mathbf{Q}_p(j))) & \\
 & & 0 & &
\end{tikzcd}$$

The first two rows are exact by Lemma 4.27 and Lemma 4.38, respectively. The middle column is exact by formula (5.17). We claim that the third column is exact, the map f_1 is injective, and the map f_2 is surjective. Indeed, by Lemma 5.15, Lemma 4.24, and Lemma 4.36, it suffices to show that the canonical map

$$H^{i-2}(\mathscr{G}_K, \tfrac{\mathscr{O}(D_C)}{C}(j-1)) \to H^{i-2}(\mathscr{G}_K, \tfrac{\mathscr{O}(Y_C)}{C}(j-1))$$

is injective. But, by the generalized Tate's formula (3.16), this map is either a map between trivial objects or is isomorphic (for $j=1$ and $i=2,3$) to the canonical map $\mathscr{O}(D)/K \to \mathscr{O}(Y)/K$. And that map is clearly injective.

The above discussion shows that we have an isomorphism:

$$H^i_c(D, \mathbf{Q}_p(j)) \xrightarrow{\sim} H^{i-2}(\mathscr{G}_K, H^2_c(D_C, \mathbf{Q}_p(j))), \quad \text{for all } i \in \mathbf{N}, j \in \mathbf{Z}. \tag{5.19}$$

Now we evoke Lemma 5.15. $\square$

5.5.2. *Open annulus.* Let A be an open annulus over K. We start with the geometric cohomology.

Lemma 5.20. (Geometric cohomology)

(1) *Let $i \in \mathbf{N}, j \in \mathbf{Z}$. The canonical maps $H^i(A_C, \mathbf{Q}_p(j)) \to H^i(\partial A_C, \mathbf{Q}_p(j))$ are injective. Hence we have $\mathscr{G}_K$-equivariant isomorphisms*

$$\begin{aligned} &H^i_c(A_C, \mathbf{Q}_p(j)) \xleftarrow{\sim} H^{i-1}(\partial A_C, \mathbf{Q}_p(j))/H^{i-1}(A_C, \mathbf{Q}_p(j)),\\ &H^i_c(A_C, \mathbf{Q}_p(j)) = 0, \quad \textit{if } i > 2. \end{aligned}$$

(2) *We have a $\mathscr{G}_K$-equivariant isomorphism and a (split) exact sequence:*

$$\begin{aligned} &H^1_c(A_C, \mathbf{Q}_p(j)) \simeq \mathbf{Q}_p(j), \\ 0 \to \tfrac{\mathscr{O}(\partial A_C)}{\mathscr{O}(A_C)\oplus C}(j-1) \to &H^2_c(A_C, \mathbf{Q}_p(j)) \to \mathbf{Q}_p(j-1) \to 0. \end{aligned} \tag{5.21}$$

Proof. This follows from the description of the groups involved in (4.31), (4.32) and Corollary 4.40. $\square$

Now we pass to arithmetic cohomology. The canonical maps

$$H^i(A, \mathbf{Q}_p(j)) \to H^i(\partial A, \mathbf{Q}_p(j))$$

are injective. This follows from the description of the groups involved in Lemma 4.34 and Corollary 4.41. Hence we have an isomorphism

$$H^i_c(A, \mathbf{Q}_p(j)) \xleftarrow{\sim} H^{i-1}(\partial A, \mathbf{Q}_p(j))/H^{i-1}(A, \mathbf{Q}_p(j)). \tag{5.22}$$

Lemma 5.23. (Arithmetic Cohomology) *Let $i \in \mathbf{N}$, $j \in \mathbf{Z}$. There is an exact sequence*

$$0 \to H^{i-1}(\mathscr{G}_K, H^1_c(A_C, \mathbf{Q}_p(j))) \to H^i_c(A, \mathbf{Q}_p(j)) \to H^{i-2}(\mathscr{G}_K, H^2_c(A_C, \mathbf{Q}_p(j))) \to 0.$$

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & H^{i-1}(\mathscr{G}_K, \mathbf{Q}_p(j)) \arrow[r] \arrow[d] & H^{i-1}(A, \mathbf{Q}_p(j)) \arrow[r] \arrow[d] & H^{i-2}(\mathscr{G}_K, H^1(A_C, \mathbf{Q}_p(j))) \arrow[r] \arrow[d, "f_1"] & 0 \\
0 \arrow[r] & H^{i-1}(\mathscr{G}_K, \mathbf{Q}_p(j)) \arrow[r] \arrow[d] & H^{i-1}(\partial A, \mathbf{Q}_p(j)) \arrow[r] \arrow[d] & H^{i-2}(\mathscr{G}_K, H^1(\partial A_C, \mathbf{Q}_p(j))) \arrow[r] \arrow[d, "f_2"] & 0 \\
 & H^{i-1}(\mathscr{G}_K, H^1(A_C, \mathbf{Q}_p(j))) \arrow[r] \arrow[d] & H^i_c(A, \mathbf{Q}_p(j)) \arrow[r] \arrow[d] & H^{i-2}(\mathscr{G}_K, H^2_c(A_C, \mathbf{Q}_p(j))) & \\
 & 0 & 0 & &
\end{tikzcd}$$

The first two rows are exact by Lemma 4.34 and Corollary 4.41, respectively. The middle column is exact by formula (5.22). The first column is exact by the same formula and the fact that the canonical map $H^0(A_C, \mathbf{Q}_p(j)) \to H^0(\partial A_C, \mathbf{Q}_p(j))$ is $\mathscr{G}_K$-equivariantly split.

It suffices now to show that the third column is exact, the map f_1 is injective, and the map f_2 is surjective. To see that, use Lemma 5.20 and note that, by Lemma 4.30 and Corollary 4.40, the map f_1 can be rewritten as the canonical map

$$H^{i-2}(\mathscr{G}_K, (\tfrac{\mathscr{O}(A_C)}{C} \oplus \mathbf{Q}_p)(j-1)) \to H^{i-2}(\mathscr{G}_K, (\tfrac{\mathscr{O}(Y_C)}{C} \oplus \mathbf{Q}_p)^{\oplus 2}(j-1)),$$

which, by the generalized Tate's formula (3.16), is either a map between trivial objects or is isomorphic (for $j = 1$ and $i = 2, 3$) to the canonical map

$$\tfrac{\mathscr{O}(A)}{K} \oplus \mathbf{Q}_p \to \big(\tfrac{\mathscr{O}(Y)}{K} \oplus \mathbf{Q}_p\big)^{\oplus 2}.$$

And that map is clearly injective. We note that this also proves that the map f_2 is surjective, as wanted. □

6. Trace maps and pairings

In this chapter we will discuss pro-étale and coherent trace maps and pairings.

6.1. **Pro-étale trace maps.** We start with pro-étale trace maps.

Proposition 6.1. *Let X be a smooth Stein variety, a smooth dagger affinoid, or a proper variety over K of dimension 1. Assume that it is geometrically irreducible. Then*

(1) There exists a natural geometric trace map

$$\mathrm{Tr}_{X_C} : H^2_c(X_C, \mathbf{Q}_p(1)) \to \mathbf{Q}_p. \tag{6.2}$$

It is an isomorphism if X is proper.

(2) There exists a natural arithmetic trace map

$$\mathrm{Tr}_X : H^4_c(X, \mathbf{Q}_p(2)) \to \mathbf{Q}_p. \tag{6.3}$$

It is an isomorphism.

(3) The above trace maps are functorial for open immersions and compatible with the Hyodo-Kato and de Rham trace maps.

Proof. In the case X is partially proper, the arithmetic trace map is defined using the geometric trace map and the Galois cohomology trace map

$$\mathrm{Tr}_X : H^4_c(X, \mathbf{Q}_p(2)) \simeq H^2(\mathscr{G}_K, H^2_c(X, \mathbf{Q}_p(2))) \xrightarrow{\mathrm{Tr}_{X_C}(1)} H^2(\mathscr{G}_K, \mathbf{Q}_p(1)) \xrightarrow{\mathrm{Tr}_K} \mathbf{Q}_p.$$

It was shown to be an isomorphism in [1, Sec. 8.3].

In the case X is a dagger affinoid, the arithmetic trace map is defined as the composition

$$\mathrm{Tr}_X : \quad H^4_c(X, \mathbf{Q}_p(2)) \xrightarrow{\sim} \lim_h H^4_c(X^0_h, \mathbf{Q}_p(2)) \xrightarrow{\lim_h \mathrm{Tr}_{X^0_h}} \mathbf{Q}_p.$$

Here the first isomorphism follows from the fact that $\mathrm{R}^1 \lim_h H^3_c(X^0_h, \mathbf{Q}_p(2)) = 0$, which was shown in the proof of Proposition 5.14. □

We will also need a derived version of the arithmetic trace map:

$$\mathrm{Tr}_X : \quad \mathrm{R}\Gamma_c(X, \mathbf{Q}_p(2))[4] \to \mathbf{Q}_p.$$

We define it as the composition

$$\mathrm{R}\Gamma_c(X, \mathbf{Q}_p(2))[4] \xleftarrow{\sim} (\tau_{\leq 4}\mathrm{R}\Gamma_c(X, \mathbf{Q}_p(2)))[4] \xrightarrow{\mathrm{can}} H^4_c(X, \mathbf{Q}_p(2)) \xrightarrow{\mathrm{Tr}_X} \mathbf{Q}_p.$$

6.2. **Pro-étale pairings for partially proper varieties.** Let X be a partially proper rigid anaytic variety over K, C. Cup product on pro-étale cohomology induces pairings

$$\mathrm{R}\Gamma(X,\mathbf{Q}_p)\otimes^{\mathrm{L}\square}_{\mathbf{Q}_p}\mathrm{R}\Gamma(\partial X,\mathbf{Q}_p)\to\mathrm{R}\Gamma(\partial X,\mathbf{Q}_p)$$

as the composition

$$\begin{aligned}\mathrm{R}\Gamma(X,\mathbf{Q}_p)\otimes^{\mathrm{L}\square}_{\mathbf{Q}_p}\mathrm{R}\Gamma(\partial X,\mathbf{Q}_p) &= \mathrm{R}\Gamma(X,\mathbf{Q}_p)\otimes^{\mathrm{L}\square}_{\mathbf{Q}_p}(\mathrm{colim}_Z\,\mathrm{R}\Gamma(X\setminus Z,\mathbf{Q}_p))\\ &\stackrel{\sim}{\leftarrow}\mathrm{colim}_Z(\mathrm{R}\Gamma(X,\mathbf{Q}_p)\otimes^{\mathrm{L}\square}_{\mathbf{Q}_p}\mathrm{R}\Gamma(X\setminus Z,\mathbf{Q}_p))\xrightarrow{\mathrm{colim}_Z\cup}\mathrm{colim}_Z\,\mathrm{R}\Gamma(X\setminus Z,\mathbf{Q}_p)\\ &=\mathrm{R}\Gamma(\partial X,\mathbf{Q}_p).\end{aligned}$$

These pairings are compatible with the pairings

$$\mathrm{R}\Gamma(X,\mathbf{Q}_p)\otimes^{\mathrm{L}\square}_{\mathbf{Q}_p}\mathrm{R}\Gamma(X,\mathbf{Q}_p)\to\mathrm{R}\Gamma(X,\mathbf{Q}_p).$$

They yield (Galois equivariant over C) pairings

$$\mathrm{R}\Gamma(X,\mathbf{Q}_p)\otimes^{\mathrm{L}\square}_{\mathbf{Q}_p}\mathrm{R}\Gamma_c(X,\mathbf{Q}_p)\to\mathrm{R}\Gamma_c(X,\mathbf{Q}_p),\tag{6.4}$$

which are compatible with the passage from K to C.

Let now X be partially proper and as in Proposition 6.1. The (twisted) pairing (6.4) composed with the trace map (6.2) yields a geometric pairing:

$$\mathrm{R}\Gamma(X_C,\mathbf{Q}_p(j))\otimes^{\mathrm{L}\square}_{\mathbf{Q}_p}\mathrm{R}\Gamma_c(X_C,\mathbf{Q}_p(1-j))[2]\to\mathbf{Q}_p.$$

Similarly, using the trace map (6.3), we obtain an arithmetic pairing

$$\mathrm{R}\Gamma(X,\mathbf{Q}_p(j))\otimes^{\mathrm{L}\square}_{\mathbf{Q}_p}\mathrm{R}\Gamma_c(X,\mathbf{Q}_p(2-j))[4]\to\mathbf{Q}_p.$$

These pairings are compatible and are also compatible with pro-étale pairings on $\mathrm{R}\Gamma(X_C,\mathbf{Q}_p(j))$ and $\mathrm{R}\Gamma(X,\mathbf{Q}_p(j))$, respectively.

6.3. **Pro-étale pairings for dagger affinoids.** Let X be a smooth dagger affinoid over K, C. Cup product on pro-étale cohomology

$$\mathrm{R}\Gamma(X,\mathbf{Q}_p)\otimes^{\mathrm{L}\square}_{\mathbf{Q}_p}\mathrm{R}\Gamma_c(X,\mathbf{Q}_p)\to\mathrm{R}\Gamma_c(X,\mathbf{Q}_p)\tag{6.5}$$

is defined by the composition

$$\begin{aligned}\mathrm{R}\Gamma(X,\mathbf{Q}_p)\otimes^{\mathrm{L}\square}_{\mathbf{Q}_p}\mathrm{R}\Gamma_c(X,\mathbf{Q}_p) &= (\mathrm{colim}_h\,\mathrm{R}\Gamma(X_h,\mathbf{Q}_p))\otimes^{\mathrm{L}\square}_{\mathbf{Q}_p}(\mathrm{colim}_n\,\mathrm{R}\Gamma_{\widehat{X}}(X_n,\mathbf{Q}_p))\\ &=\mathrm{colim}_{h,n}(\mathrm{R}\Gamma(X_h,\mathbf{Q}_p)\otimes^{\mathrm{L}\square}_{\mathbf{Q}_p}\mathrm{R}\Gamma_{\widehat{X}}(X_n,\mathbf{Q}_p))\\ &\stackrel{\sim}{\to}\mathrm{colim}_h(\mathrm{R}\Gamma(X_h,\mathbf{Q}_p)\otimes^{\mathrm{L}\square}_{\mathbf{Q}_p}\mathrm{R}\Gamma_{\widehat{X}}(X_h,\mathbf{Q}_p))\\ &\xrightarrow{\mathrm{colim}_h\cup}\mathrm{colim}_h\,\mathrm{R}\Gamma_{\widehat{X}}(X_h,\mathbf{Q}_p)=\mathrm{R}\Gamma_c(X,\mathbf{Q}_p).\end{aligned}$$

Here $\{X_h\}_{h\in\mathbf{N}}$ is the dagger presentation of X.

These pairings are Galois equivariant over C and compatible with the passage from K to C.

The (twisted) pairing (6.5) composed with the trace map (6.2) yields a geometric pairing:

$$\mathrm{R}\Gamma(X_C,\mathbf{Q}_p(j))\otimes^{\mathrm{L}\square}_{\mathbf{Q}_p}\mathrm{R}\Gamma_c(X_C,\mathbf{Q}_p(1-j))[2]\to\mathbf{Q}_p.$$

Similarly, using the trace map (6.3), we obtain an arithmetic pairing

$$\mathrm{R}\Gamma(X,\mathbf{Q}_p(j))\otimes^{\mathrm{L}\square}_{\mathbf{Q}_p}\mathrm{R}\Gamma_c(X,\mathbf{Q}_p(2-j))[4]\to\mathbf{Q}_p.$$

These pairings are compatible and are also compatible with pro-étale pairings on $\mathrm{R}\Gamma(X_C,\mathbf{Q}_p(j))$ and $\mathrm{R}\Gamma(X,\mathbf{Q}_p(j))$, respectively.

6.4. **Coherent pairings.** We will list now the coherent pairings that we will use.

6.4.1. *Ghost circle.* Let D be an open disc over K; let $Y := \partial D$ be the boundary of D, a ghost circle. Let $Y_C := Y \times C$. The ring $\mathscr{O}(Y)$ (resp. $\mathscr{O}(Y_C)$) is the Robba ring with coefficients in K (resp. C). Topologically, $\mathscr{O}(Y)$ is a direct sum of a nuclear Fréchet space and a space of compact type (both over $\mathbf{Q}_p$).

The map

$$(f,g) \mapsto \mathrm{Tr}_K \mathrm{res}(f dg) \tag{6.6}$$

induces a perfect pairing[21]

$$\cup : \mathscr{O}(Y)/K \otimes^{\square}_{\mathbf{Q}_p} \mathscr{O}(Y)/K \to \mathbf{Q}_p \tag{6.7}$$

between the solid $\mathbf{Q}_p$-vector spaces $\mathscr{O}(Y)/K$ and $\mathscr{O}(Y)/K$.

Similarly, for $L = K, C$, the map

$$(f,g) \mapsto \mathrm{res}(f dg) \tag{6.8}$$

induces a perfect pairing

$$\cup : \mathscr{O}(Y_L)/L \otimes^{\square}_{L} \mathscr{O}(Y_L)/L \to L \tag{6.9}$$

between the solid L-vector spaces $\mathscr{O}(Y_L)/L$ and $\mathscr{O}(Y_L)/L$.

Remark 6.10. We can see one of the copies of $\mathscr{O}(Y_L)/L$ as the space $\Omega^1(Y_L)_0$ of differential forms with residue equal to 0 (via $h \mapsto dh$). The map $(f,\omega) \mapsto \mathrm{res}(f\omega)$ induces a perfect duality between the L-spaces $\mathscr{O}(Y_L)$ and $\Omega^1(Y_L)$ and $\Omega^1(Y_L)_0$ is exactly the orthogonal of $L \subset \mathscr{O}(Y_L)$.

6.4.2. *Open disc.* Let D and Y be as above. We see easily that $\mathscr{O}(Y)/\mathscr{O}(D)$ is the dual of $\mathscr{O}(D)/K$ for the pairing (6.6), i.e., that we have a perfect pairing

$$\cup : \mathscr{O}(Y)/\mathscr{O}(D) \otimes^{\square}_{\mathbf{Q}_p} \mathscr{O}(D)/K \to \mathbf{Q}_p.$$

Similarly, we check that $\mathscr{O}(Y_L)/\mathscr{O}(D_L)$ is the dual of $\mathscr{O}(D_L)/L$ for the pairing (6.8), i.e., that we have a perfect pairing

$$\cup : \mathscr{O}(Y_L)/\mathscr{O}(D_L) \otimes^{\square}_{L} \mathscr{O}(D_L)/L \to L.$$

This pairing can be thought of as Serre duality pairing

$$\cup : H^1_c(D_L, \mathscr{O}) \otimes^{\square}_{L} H^0(D_L, \Omega^1) \to L.$$

This is because we have isomorphisms (the second is just $d : \mathscr{O} \to \Omega^1$)

$$\mathscr{O}(Y_L)/\mathscr{O}(D_L) \xrightarrow{\sim} H^1_c(D_L, \mathscr{O}), \quad \mathscr{O}(D_L)/L \xrightarrow{\sim} H^0(D_L, \Omega^1).$$

6.4.3. *Open annulus.* Let A be an open annulus over K; let ∂A be the boundary of A, a disjoint union $Y_a \sqcup Y_b$ of two ghost circles. We see easily that $\mathscr{O}(\partial A)/(\mathscr{O}(A) \oplus K)$ is the dual of $\mathscr{O}(A)/K$ for the pairing (6.6) taken twice and followed by the addition map $\mathbf{Q}_p^{\oplus 2} \xrightarrow{+} \mathbf{Q}_p$. That is, that we have a perfect pairing

$$\cup : \mathscr{O}(\partial A)/(\mathscr{O}(A) \oplus K) \otimes^{\square}_{\mathbf{Q}_p} \mathscr{O}(A)/K \to \mathbf{Q}_p.$$

Similarly, we check that $\mathscr{O}(\partial A_L)/(\mathscr{O}(A_L) \oplus L)$ is the dual of $\mathscr{O}(A_L)/L$ for the pairing (6.8), i.e., that we have a perfect pairing

$$\cup : \mathscr{O}(\partial A_L)/(\mathscr{O}(A_L) \oplus L) \otimes^{\square}_{L} \mathscr{O}(A_L)/L \to L.$$

This pairing can be seen as induced by the Serre duality pairing

$$\cup : H^1_c(A_L, \mathscr{O}) \otimes^{\square}_{L} H^0(A_L, \Omega^1) \to L.$$

[21] We used here the fact that, for nuclear Fréchet spaces and spaces of compact type, the passage between locally convex topological vector spaces and solid vector spaces works well on the level of tensor products and Homs, see Section 2.2.5.

This is because we have an isomorphism

$$\mathscr{O}(\partial A_L)/\mathscr{O}(A_L) \xrightarrow{\sim} H^1_c(A_L, \mathscr{O})$$

and the exact sequence

$$0 \to \mathscr{O}(A_L)/L \xrightarrow{d} H^0(A_L, \Omega^1) \to L\tfrac{dz}{z} \to 0.$$

7. Poincaré duality for a ghost circle

Let K be a finite extension of $\mathbf{Q}_p$. We will prove in this chapter arithmetic Poincaré duality for a ghost circle Y over K that will be essential in later computations. The numerology suggests that Y is a "proper" variety of total dimension $\frac{3}{2}$, hence Y_C is of dimension $\frac{1}{2}$.

7.1. Arithmetic duality theorem.

We start with a proof that assumes an Explicit Reciprocity Law (which will be proved in the next section).

Theorem 7.1. (Arithmetic duality) *Let Y be a ghost circle over K.*

(i) *There is a trace map isomorphism*

$$\mathrm{Tr}_Y:\ H^3(Y, \mathbf{Q}_p(2)) \xrightarrow{\sim} \mathbf{Q}_p.$$

(ii) *Let $i, j \in \mathbf{Z}$. The pairing*

$$\cup:\quad H^i(Y, \mathbf{Q}_p(j)) \otimes^{\square}_{\mathbf{Q}_p} H^{3-i}(Y, \mathbf{Q}_p(2-j)) \xrightarrow{\cup} H^3(Y, \mathbf{Q}_p(2)) \xrightarrow[\sim]{\mathrm{Tr}_Y} \mathbf{Q}_p \tag{7.2}$$

is a perfect duality, i.e., we have an induced isomorphism

$$\gamma_{Y,i}:\quad H^i(Y, \mathbf{Q}_p(j)) \xrightarrow{\sim} H^{3-i}(Y, \mathbf{Q}_p(2-j))^*.$$

Proof. Define geometric and arithmetic trace maps as follows:

$$\mathrm{Tr}_{Y_C} : H^1(Y_C, \mathbf{Q}_p(1)) \to \mathbf{Q}_p,$$

$$\mathrm{Tr}_Y : H^3(Y, \mathbf{Q}_p(2)) \xrightarrow{\sim} H^2(\mathscr{G}_K, H^1(Y_C, \mathbf{Q}_p(2))) \xrightarrow[\sim]{\mathrm{Tr}_{Y_C}(1)} H^2(\mathscr{G}_K, \mathbf{Q}_p(1)) \xrightarrow[\sim]{\mathrm{Tr}_K} \mathbf{Q}_p.$$

Here, the first trace map comes from the exact sequence in (4.37) and $\mathrm{Tr}_{Y_C}(1)$ is an isomorphism because $H^2(\mathscr{G}_K, \frac{\mathscr{O}(Y_C)}{C}) = 0$. This proves (i); let us turn to (ii).

• *Filtration on cohomology.* Let $i, j \in \mathbf{Z}$. There exists an ascending filtration on $H^i(Y, \mathbf{Q}_p(j))$:

$$F^2_{i,j} = H^i(Y, \mathbf{Q}_p(j)) \supset F^1_{i,j} \supset F^0_{i,j} \supset F^{-1}_{i,j} = 0, \tag{7.3}$$

such that we have isomorphisms

$$\begin{aligned} F^2_{i,j}/F^1_{i,j} &\simeq H^{i-1}(\mathscr{G}_K, \mathbf{Q}_p(j-1)),\\ F^1_{i,j}/F^0_{i,j} &\simeq H^{i-1}(\mathscr{G}_K, \tfrac{\mathscr{O}(Y_C)}{C}(j-1)),\\ F^0_{i,j}/F^{-1}_{i,j} &\simeq H^{i}(\mathscr{G}_K, \mathbf{Q}_p(j)). \end{aligned}$$

This follows from Section 4.4, Lemma 4.38, and Lemma 4.36.

• *Identification of pairings on graded pieces.* Assume the following:

Theorem 7.4. (Explicit Reciprocity Law)

(i) *The pairing (7.2) is compatible with the above filtration. In particular $F^a_{i,j}$ and $F^b_{3-i,2-j}$ are orthogonal if $a + b \leq 1$.*

(ii) *On the associated grading the pairing (7.2) yields a pairing induced by the Galois cohomology pairing and pairing (6.9).*

From claim (i), we obtain the following commutative diagram with exact rows (all the vertical maps are induced from pro-étale cup products and the trace Tr_Y):
(7.5)

$$\begin{tikzcd} 0 \arrow[r] & H^{i-1}(\mathscr{G}_K, \frac{\mathscr{O}(Y_C)}{C}(j-1)) \arrow[r] \arrow[d, "\wr"] & H^{i-1}(\mathscr{G}_K, H^1(Y_C, \mathbf{Q}_p(j))) \arrow[r] \arrow[d, "\beta_{Y,i}"] & H^{i-1}(\mathscr{G}_K, \mathbf{Q}_p(j-1)) \arrow[r] \arrow[d, "\wr"] & 0 \\ 0 \arrow[r] & H^{2-i}(\mathscr{G}_K, \frac{\mathscr{O}(Y_C)}{C}(1-j))^* \arrow[r] & (F^1_{3-i,2-j})^* \arrow[r] & H^{3-i}(\mathscr{G}_K, \mathbf{Q}_p(2-j))^* \arrow[r] & 0 \end{tikzcd}$$

By Theorem 7.4, claim (ii), the left and the right vertical arrows are isomorphisms. It follows that we have an isomorphism (we skipped the indices to lighten the notation)

$$\beta_{Y,i}: \quad (F^2/F^0)H^i(Y, \mathbf{Q}_p(j)) \xrightarrow{\sim} (F^1 H^{3-i}(Y, \mathbf{Q}_p(2-j)))^*.$$

Similarly, we obtain the following commutative diagram with exact rows (again, all the vertical maps are induced from pro-étale cup products and the trace Tr_Y)

$$(7.6) \quad \begin{tikzcd} 0 \arrow[r] & H^i(\mathscr{G}_K, \mathbf{Q}_p(j)) \arrow[r] \arrow[d, "\alpha_{Y,i}", "\wr"'] & H^i(Y, \mathbf{Q}_p(j)) \arrow[r] \arrow[d, "\gamma_{Y,i}"] & H^{i-1}(\mathscr{G}_K, H^1(Y_C, \mathbf{Q}_p(j))) \arrow[r] \arrow[d, "\beta_{Y,i}", "\wr"'] & 0 \\ 0 \arrow[r] & H^{2-i}(\mathscr{G}_K, \mathbf{Q}_p(1-j))^* \arrow[r] & H^{3-i}(Y, \mathbf{Q}_p(2-j))^* \arrow[r] & (F^1_{3-i,2-j})^* \arrow[r] & 0 \end{tikzcd}$$

By Theorem 7.4, the map $\alpha_{Y,i}$ is induced by the Galois pairing hence is an isomorphism. It follows that so is the map $\gamma_{Y,i}$, as wanted. $\square$

7.2. **Proof of the Explicit Reciprocity Law.** The goal of the rest of the chapter is to prove Theorem 7.4. The result is immediate for $i \geq 4$ since then all the terms are 0.

For $i = 0, 3$, by symmetry, we may assume that $i = 0$. We have

$$H^0(Y, \mathbf{Q}_p(j)) \simeq \begin{cases} \mathbf{Q}_p & \text{if } j = 0, \\ 0 & \text{if } j \neq 0; \end{cases} \quad H^3(Y, \mathbf{Q}_p(2-j)) \simeq \begin{cases} \mathbf{Q}_p & \text{if } j = 0, \\ 0 & \text{if } j \neq 0. \end{cases}$$

Here, to compute, H^3 we have used Lemma 4.38 and Lemma 4.36. Now we need to study the pairing

$$\cup: \quad H^0(Y, \mathbf{Q}_p) \otimes^{\square}_{\mathbf{Q}_p} H^3(Y, \mathbf{Q}_p(2)) \to H^3(Y, \mathbf{Q}_p(2)) \xrightarrow[\sim]{\mathrm{Tr}_Y} \mathbf{Q}_p.$$

We have

$$\begin{aligned} &F^0_{0,0} = F^1_{0,0} = F^2_{0,0} = H^0(Y, \mathbf{Q}_p), \\ &F^0_{3,2} = F^1_{3,2} = 0, \quad F^2_{0,0} = H^3(Y, \mathbf{Q}_p(2)). \end{aligned}$$

Claim (i) of Theorem 7.4 follows immediately. Claim (ii) is easy to check by following the isomorphisms appearing in Lemma 4.38, Lemma 4.36 and the definition of the trace map Tr_Y, and by compatibility of the Hochschild-Serre spectral sequence with products. (The Hochschild-Serre spectral sequence for Y is obtained by taking colim of the Hochschild-Serre spectral sequences for annuli (4.33). The fact that colim commutes with Galois cohomology is proved as in the proof of Lemma 4.36.)

So it remains to look at $i = 1, 2$ and, by symmetry, we may assume that $i = 1$. That is, we are studying the pairing

$$(7.7) \qquad \cup: \quad H^1(Y, \mathbf{Q}_p(j)) \otimes^{\square}_{\mathbf{Q}_p} H^2(Y, \mathbf{Q}_p(2-j)) \to H^3(Y, \mathbf{Q}_p(2)) \xrightarrow[\sim]{\mathrm{Tr}_Y} \mathbf{Q}_p.$$

7.3. **Descent to $\widehat{K}_\infty$.** Since $\overline{\mathbf{Q}}_p/K_\infty$ is almost étale, we can compute the pro-étale cohomology of $Y_K := Y$ using $Y_{\widehat{K}_\infty}$ and the latter is computed via syntomic methods as in [16] or in [21].

7.3.1. $(\partial,\varphi,\gamma)$-*Koszul complexes.* Let $\widetilde{Y}^{[u,v]}=Y_{\mathbf{Q}_p}\times U^{[u,v]}$ (see section 3.2.1). Then we have

$$\widetilde{\mathscr{O}}^{[u,v]}:=\mathscr{O}(\widetilde{Y}^{[u,v]})\simeq\mathscr{O}(Y_{\mathbf{Q}_p})\widehat{\otimes}\mathbf{B}_{K_\infty}^{[u,v]},\quad \widetilde{\mathscr{O}}^{[u,v]}/t\simeq\mathscr{O}(Y_{\widehat{K}_\infty}).$$

The algebra $\mathscr{O}(Y_{\mathbf{Q}_p})$ is a direct sum of a nuclear Fréchet space and a space of compact type and $\mathbf{B}_{K_\infty}^{[u,v]}$ is a Banach space. Let $j\in\mathbf{Z}$. Let us choose a uniformizer T of $\mathscr{O}(Y_{\mathbf{Q}_p})$ and define Frobenius φ by sending T to T^p and let $\partial=T\frac{d}{dT}$. We denote by

$$\mathrm{Kos}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}(j))\quad\text{and}\quad\mathrm{Kos}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}(j))^{\Delta_K}\tag{7.8}$$

the total complex of the double complex

$$\begin{array}{ccccc}\widetilde{\mathscr{O}}^{[u,v]}(j) & \xrightarrow{(t\partial,\varphi-1)} & \widetilde{\mathscr{O}}^{[u,v]}(j-1)\oplus\widetilde{\mathscr{O}}^{[u,v/p]}(j) & \xrightarrow{(\varphi-1)-t\partial} & \widetilde{\mathscr{O}}^{[u,v/p]}(j-1)\\ \big\downarrow{\scriptstyle\gamma_K-1} & & \big\downarrow{\scriptstyle\gamma_K-1}\qquad\big\downarrow{\scriptstyle\gamma_K-1} & & \big\downarrow{\scriptstyle\gamma_K-1}\\ \widetilde{\mathscr{O}}^{[u,v]}(j) & \xrightarrow{(t\partial,\varphi-1)} & \widetilde{\mathscr{O}}^{[u,v]}(j-1)\oplus\widetilde{\mathscr{O}}^{[u,v/p]}(j) & \xrightarrow{(\varphi-1)-t\partial} & \widetilde{\mathscr{O}}^{[u,v/p]}(j-1)\end{array}\tag{7.9}$$

and the complex obtained by taking fixed points by Δ_K of each of its terms[22]. Since they are based on Fontaine-Messing syntomic cohomology (see Remark 7.12), we will call the Koszul complexes (7.8) *FM-Koszul complexes.*

They sometimes appear in a different form

$$\mathrm{Kos}^{\mathrm{HK}}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}(j)),\quad\text{and}\quad\mathrm{Kos}^{\mathrm{HK}}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}(j))^{\Delta_K},\tag{7.10}$$

which we will call *HK-Koszul complexes* as they are based on Hyodo-Kato syntomic cohomology. The first one is the total complex of the double complex

$$\begin{array}{ccccc}\widetilde{\mathscr{O}}^{[u,v]}(j-1) & \xrightarrow{(\partial,\frac{\varphi}{p}-1,\mathrm{can})} & \widetilde{\mathscr{O}}^{[u,v]}(j-1)\oplus\widetilde{\mathscr{O}}^{[u,v/p]}(j-1)\oplus(\widetilde{\mathscr{O}}^{[u,v]}/t)(j-1) & \xrightarrow{(\varphi-1)-\partial+0} & \widetilde{\mathscr{O}}^{[u,v/p]}(j-1)\\ \big\downarrow{\scriptstyle\gamma_K-1} & & \big\downarrow{\scriptstyle\gamma_K-1}\qquad\big\downarrow{\scriptstyle\gamma_K-1}\qquad\big\downarrow{\scriptstyle\gamma_K-1} & & \big\downarrow{\scriptstyle\gamma_K-1}\\ \widetilde{\mathscr{O}}^{[u,v]}(j-1) & \xrightarrow{(\partial,\frac{\varphi}{p}-1,\mathrm{can})} & \widetilde{\mathscr{O}}^{[u,v]}(j-1)\oplus\widetilde{\mathscr{O}}^{[u,v/p]}(j-1)\oplus(\widetilde{\mathscr{O}}^{[u,v]}/t)(j-1) & \xrightarrow{(\varphi-1)-\partial+0} & \widetilde{\mathscr{O}}^{[u,v/p]}(j-1)\end{array}$$

The two Koszul complexes are related by a quasi-isomorphism in $\mathscr{D}(\mathbf{Q}_{p,\square})$

$$\beta_j:\mathrm{Kos}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}(j))\xrightarrow{\sim}\mathrm{Kos}^{\mathrm{HK}}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}(j))\tag{7.11}$$

given by the maps $(t,\mathrm{Id}\oplus t\oplus 0,\mathrm{Id})$ in the top and bottom rows (same for Δ_K-fixed points).

Remark 7.12. (*Relation to syntomic cohomology.*)

(i) One can easily show that the horizontal complexes in (7.9) compute the syntomic cohomology of $Y_{\widehat{K}_\infty}$ (more exactly, $\mathrm{R\Gamma}_{\mathrm{syn}}(Y_{\widehat{K}_\infty},1)(j-1)$; the twist $(j-1)$ does not play a role at the level of K_∞ but intervenes in the computation of the arithmetic cohomology). The complex $\mathrm{Kos}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}(j))$ is then given by the mapping fiber

$$[\ \mathrm{R\Gamma}_{\mathrm{syn}}(Y_{\widehat{K}_\infty},1)(j-1)\xrightarrow{\gamma_K-1}\mathrm{R\Gamma}_{\mathrm{syn}}(Y_{\widehat{K}_\infty},1)(j-1)\ ].\tag{7.13}$$

By applying $\mathrm{Res}_{p^{-n}\mathbf{Z}_p}$, for n big enough, we obtain a quasi-isomorphic complex in which $\mathbf{B}_{K_\infty}^{[u,v]}$ is replaced by $\mathbf{B}_{K_n}^{[u,v]}$ and which looks like $\mathrm{R\Gamma}_{\mathrm{syn}}(Y_K,1)(j-1)$ except for the arithmetic variable which is treated a little bit differently (there is an action of the entire Γ_K and not just of its Lie algebra).

[22]In what follows, a superscript Δ_K means taking fixed points by Δ_K of each terms; we don't need to take derived fixed points since Δ_K is finite and the modules are L-vector spaces. All the statements that follow continue to hold for fixed points by Δ_K but we will not state them explicitly.

(ii) More precisely, the double complex (7.8) can be rewritten in the following way:

$$\begin{tikzcd} F^1\widetilde{\mathscr{O}}^{[u,v]}(j-1) \arrow[r, "(d,\frac{\varphi}{p}-1)"] \arrow[d, "\gamma_K-1"] & \widetilde{\Omega}^{[u,v]}(j-1)\oplus\widetilde{\mathscr{O}}^{[u,v/p]}(j-1) \arrow[r, "(\frac{\varphi}{p}-1)-d"] \arrow[d, "\gamma_K-1"] & \widetilde{\Omega}^{[u,v/p]}(j-1) \arrow[d, "\gamma_K-1"] \\ F^1\widetilde{\mathscr{O}}^{[u,v]}(j-1) \arrow[r, "(d,\frac{\varphi}{p}-1)"] & \widetilde{\Omega}^{[u,v]}(j-1)\oplus\widetilde{\mathscr{O}}^{[u,v/p]}(j-1) \arrow[r, "(\frac{\varphi}{p}-1)-d"] & \widetilde{\Omega}^{[u,v/p]}(j-1). \end{tikzcd} \tag{7.14}$$

We remind the reader that $F^1\widetilde{\mathscr{O}}^{[u,v]}=t\widetilde{\mathscr{O}}^{[u,v]}$, and that multiplication by t is equivalent to a Tate twist by (1). An isomorphism from the complex (7.8) to (7.14) can be given by the map

$$\begin{bmatrix} a & x & y & z \\ a' & x' & y' & z' \end{bmatrix} \Rightarrow \begin{bmatrix} ta & x\frac{dT}{T} & ty & z\frac{dT}{T} \\ ta' & x'\frac{dT}{T} & ty' & z'\frac{dT}{T} \end{bmatrix}$$

(iii) We have chosen to twist everything in (i) by (1) but one could twist by (r) for any $r\geq 1$. That is, take F^r, $F^{r-1}\Omega^1$, $\frac{\varphi}{p^r}$, etc.

7.3.2. *Products on mapping fibers.* We will recall here the well-known formulas for products on mapping fibers that we will need (for details, see, for example, [26, Prop. 3.1]).

Let $A_i^\bullet$, $C_i^\bullet$ complexes of condensed $\mathbf{Q}_p$-vector spaces (for $i=1,2,3$) and f_i,g_i morphisms of complexes $A_i^\bullet\to C_i^\bullet$. Assume that, for all α in $\mathbf{Q}_p$, there are maps

$$\cup_\alpha : A_1^\bullet\otimes_{\mathbf{Q}_p}A_2^\bullet\to A_3^\bullet \text{ and } \cup_\alpha : C_1^\bullet\otimes_{\mathbf{Q}_p}C_2^\bullet\to C_3^\bullet$$

such that the $\cup_\alpha$'s are morphisms of complexes which commute with the f_i's and g_i's, all the $\cup_\alpha$'s are homotopic and we can choose the homotopies such that they commute with the f_i's and g_i's.

If we take the mapping fiber

$$D_i^\bullet := [A_i^\bullet \xrightarrow{f_i-g_i} C_i^\bullet]$$

and, for all $\alpha\in\mathbf{Q}_p$, the products

$$\cup_\alpha : D_1^\bullet\otimes_{\mathbf{Q}_p}D_2^\bullet\to D_3^\bullet$$

can be defined (on the level of sections) by the formula

$$\gamma_1\cup_\alpha\gamma_2=(a_1\cup_\alpha a_2, c_1\cup_\alpha w_\alpha(a_2)+(-1)^{\deg(a_1)}w_{1-\alpha}(a_1)\cup_\alpha c_2), \tag{7.15}$$

where $(a_i,c_i)\in A_i^\bullet\oplus C_i^{\bullet-1}$ represents γ_i, and, for $\beta\in\mathrm{R}$, $w_\beta=\beta f_i(a_i)+(1-\beta)g_i(a_i)$.

Then:

(1) The $\cup_\alpha$'s are morphisms of complexes, which commute with the projections $D_i^\bullet\to A_i^\bullet$.
(2) The $\cup_\alpha$'s are homotopic.
(3) If $\tilde{A}_1^\bullet$, $\tilde{C}_i^\bullet$, $\tilde{f}_i$, $\tilde{\cup}_\alpha$ are another set of data as above and $A_i^\bullet\to\tilde{A}_i^\bullet$, $C_i^\bullet\to\tilde{C}_i^\bullet$ are morphisms of complexes which commute with $\cup_\alpha$ and $\tilde{\cup}_\alpha$, then the induced morphism $D_i^\bullet\to\tilde{D}_i^\bullet$ commutes with $\cup_\alpha$ and $\tilde{\cup}_\alpha$ defined by (7.15).
(4) If $g_i=0$, then the products $\cup_0,\cup_1 : D_1^\bullet\otimes_{\mathbf{Q}_p}D_2^\bullet\to D_3^\bullet$ induce products $\tilde{\cup}_0,\tilde{\cup}_1 : A_1^\bullet\otimes_{\mathbf{Q}_p}D_2^\bullet\to D_3^\bullet$ such that the following diagrams are commutative:

$$\begin{tikzcd} A_1^\bullet\otimes_{\mathbf{Q}_p}D_2^\bullet \arrow[r, "\tilde{\cup}_0"] & D_3^\bullet \\ D_1^\bullet\otimes_{\mathbf{Q}_p}D_2^\bullet \arrow[u] \arrow[ur, "\cup_0"'] & \end{tikzcd} \qquad \begin{tikzcd} D_2^\bullet\otimes_{\mathbf{Q}_p}A_1^\bullet \arrow[r, "\tilde{\cup}_1"] & D_3^\bullet \\ D_2^\bullet\otimes_{\mathbf{Q}_p}D_1^\bullet \arrow[u] \arrow[ur, "\cup_1"'] & \end{tikzcd} .$$

Explicitly, for $a_i\in A_i^\bullet$ and $c_i\in C_i^\bullet$, we have

$$a_1\tilde{\cup}_0(a_2,c_2)=(a_1\cup_0 a_2,(-1)^{\deg(a_1)}f_1(a_1)\cup_0 c_2),$$

$$(a_2,c_2)\tilde{\cup}_1 a_1=(a_2\cup_1 a_1, c_2\cup_1 f_1(a_1)).$$

7.3.3. *Structures on $(\partial,\varphi,\gamma)$-Koszul complexes.* We equip the complexes $\mathrm{Kos}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}(j))$ with the following structures:

($\bullet$) *Products.* Using (twice) the formulas from Section 7.3.2, we define cup products ($\alpha\in\mathbf{Q}_p$)

$$\cup_\alpha:\mathrm{Kos}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}(j_1))\otimes^{\square}_{\mathbf{Q}_p}\mathrm{Kos}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}(j_2))\to\mathrm{Kos}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}(j_1+j_2)).$$

($\bullet$) *Filtration.* We note that the operators ∂,γ_K do not change the exponents in the powers of T (and that the operator φ sends T to T^p). In particular, we can separate the constant term from the rest, which allows us to write

$$\mathrm{Kos}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}(j))=\mathrm{Kos}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}_0(j))\oplus\mathrm{Kos}_{\varphi,\gamma}(\mathbf{B}^{[u,v]}_{K_\infty}(j))\oplus\mathrm{Kos}_{\varphi,\gamma}(\mathbf{B}^{[u,v]}_{K_\infty}(j-1))[-1].\tag{7.16}$$

Here the subscript 0 in $\widetilde{\mathscr{O}}^{[u,v]}_0$ in the first term denotes the series for which the constant term is 0. The complex $\mathrm{Kos}_{\varphi,\gamma}(\mathbf{B}^{[u,v]}_{K_\infty}(s))$ (for $s=j,j-1$) is the one defined in Section 3.2 and is quasi-isomorphic to $\mathrm{R}\Gamma(\mathscr{G}_K,\mathbf{Q}_p(s))$ (see (3.5)).

We define an ascending filtration on the complex $\mathrm{Kos}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}(j))$:

$$\begin{aligned}F^{-1}:=0,\ \ F^0&:=\mathrm{Kos}_{\varphi,\gamma}(\mathbf{B}^{[u,v]}_{K_\infty}(j)),\\ F^1&:=\mathrm{Kos}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}_0(j))\oplus\mathrm{Kos}_{\varphi,\gamma}(\mathbf{B}^{[u,v]}_{K_\infty}(j)),\\ F^2&:=\mathrm{Kos}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}(j)).\end{aligned}$$

This filtration does not depend on the choice of T (but the splitting does): this is obvious for F^0 and, on the diagram (7.14), F^1 corresponds to the kernel of the residue map on differential forms. We have canonical quasi-isomorphisms in $\mathscr{D}(\mathbf{Q}_{p,\square})$:

$$\begin{aligned}(F^0/F^{-1})\mathrm{Kos}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}(j))&\simeq\mathrm{Kos}_{\varphi,\gamma}(\mathbf{B}^{[u,v]}_{K_\infty}(j)),\\ (F^1/F^{0})\mathrm{Kos}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}(j))&\simeq\mathrm{Kos}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}_0(j)),\\ (F^2/F^{1})\mathrm{Kos}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}(j))&\simeq\mathrm{Kos}_{\varphi,\gamma}(\mathbf{B}^{[u,v]}_{K_\infty}(j-1))[-1].\end{aligned}$$

($\bullet$) *Trace map.* We define the trace map

$$\mathrm{Tr}_{\mathrm{Kos}}:\quad H^3\mathrm{Kos}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}(2))\xrightarrow{\sim}\mathbf{Q}_p$$

as the composition

$$\begin{aligned}\mathrm{Tr}_{\mathrm{Kos}}:\quad H^3\mathrm{Kos}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}(2))&\xrightarrow{\sim}H^3\big((F^2/F^1)\mathrm{Kos}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}(1))\big)\\ &\simeq H^2\mathrm{Kos}_{\varphi,\gamma}(\mathbf{B}^{[u,v]}_{K_\infty}(1))\xrightarrow{\sim}\mathbf{Q}_p,\end{aligned}$$

where the last map is $\frac{1}{|\Delta_K|}\mathrm{res}_\pi\circ\mathrm{Tr}_{K_\infty/F_\infty}$ (see Proposition 3.11).

7.3.4. *Quasi-isomorphism with pro-étale cohomology.* Recall that, if we start with the complex computing the cohomology of the π_1 via perfectoid methods, we get the same complex[23] (but with slightly different period rings) as $\mathrm{Kos}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}(j))$ and with $t\partial$ replaced by $\tau-1$, where τ is the generator of the geometric part of $\mathrm{Aut}(\mathscr{O}(Y_{\widehat{K}_\infty}[T^{1/p^\infty}])/\mathscr{O}(Y))$ (this group is the semi-direct product of $\mathbf{Z}_p(1)$ and Γ_K, and τ is the generator of $\mathbf{Z}_p(1)$ given by our fixed choice of compatible system of roots of unity). Passing from $\tau-1$ to $t\partial$ corresponds to passing from $\mathbf{Z}_p(1)$ to its Lie algebra as in [16] or [21]; change of period rings is done as in [16] or [21]. It follows that we have a quasi-isomorphism in $\mathscr{D}(\mathbf{Q}_{p,\square})$

$$\alpha_j:\mathrm{Kos}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}(j))^{\Delta_K}\xrightarrow{\sim}\mathrm{R}\Gamma(Y_K,\mathbf{Q}_p(j)).\tag{7.17}$$

[23]More precisely, one needs to do it for each of the annuli converging to the ghost circle and then go to the limit.

Remark 7.18. (i) It is easy to see that the maps (7.17) and (3.5) are compatible.

(ii) The following commutative diagram[24] makes it possible to assume that $\Delta_K = 1$ in the proofs:

$$\begin{tikzcd}
\mathrm{Kos}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}(j)) \arrow[r, "\sim"] \arrow[d, shift left, "{[\Delta_K]}"] & \mathrm{R}\Gamma(Y_L, \mathbf{Q}_p(j)) \arrow[d, shift left, "\mathrm{cor}_L^K"] \\
\mathrm{Kos}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}(j))^{\Delta_K} \arrow[r, "\sim"] \arrow[u, shift left, "\mathrm{id}"] & \mathrm{R}\Gamma(Y_K, \mathbf{Q}_p(j)) \arrow[u, shift left, "\mathrm{res}_K^L"]
\end{tikzcd}$$

Lemma 7.19. *On cohomology level, the quasi-isomorphism α_j from* (7.17) *is compatible with products, filtrations, and trace maps.*

Proof. (i) *Products.* The easiest way to see this is to trace the geometric part of the quasi-isomorphism α_j via the analog of the big commutative diagram used in the proof of Theorem 7.5 in [21]. We choose r large enough (see Remark 7.12). The best path is via the top row (right-to-left), then all the way down, and then to the right along the bottom row. All the morphisms used along the way are clearly compatible with cup products. That treats the case of the cohomology of the geometric π_1.

Now we apply continuous group cohomology for the Galois group of the base field. This and the subsequent almost étale descent are clearly compatible with products. What remains is the passage from (nonhomogeneous) continuous cochains of $< \gamma_K >$ to the Koszul complexes for γ_K. But a map from the former to the latter can be given by the identity in degree 0 and the evaluation at γ_K in degree 1; this is easily checked to be compatible with products.

(ii) *Filtrations.* This is easy to see for F^0: both the pro-étale F^0 and the (φ,Γ)-module F^0 come from the corresponding complexes of a point (given by K). The required compatibility follows then easily from functoriality of the comparison map α_j (see Remark 7.18). Moreover, α_j restricted to F^0 is an isomorphism (on cohomology level).

It remains to check compatibility for F^1. Or, in light of the above, for F^1/F^0. Note that after moding out F^0 from both sides of the map α_j, the pro-étale F^1 arises from the Galois (for the group $\mathscr{G}_K$) kernel of the pro-étale residue map

$$\mathrm{res}_{\text{proét}} : H^1(Y_C, \mathbf{Q}_p(j)) \to \mathbf{Q}_p(j-1)$$

(see (7.3)). Recall that this map comes from the syntomic residue map

$$\mathrm{res}_{\mathrm{syn}}(j-1) : H^1_{\mathrm{syn}}(Y_C, 1)(j-1) \to \mathbf{B}_{\mathrm{cr}}^{+,\varphi=1}(j-1) \tag{7.20}$$

(See diagram (4.8), take $i = 1$, and note that the slope of Frobenius on the Hyodo-Kato cohomology is 1.) By changing the period rings, the Galois cohomology of the map (7.20) can be seen as the residue map from diagram (7.13) to diagram

$$\begin{tikzcd}
\mathbf{B}_{K_\infty}^{[u,v]}(j-1) \arrow[r, "\varphi-1"] \arrow[d, "\gamma_K-1"] & \mathbf{B}_{K_\infty}^{[u,v/p]}(j-1) \arrow[d, "\gamma_K-1"] \\
\mathbf{B}_{K_\infty}^{[u,v]}(j-1) \arrow[r, "\varphi-1"] & \mathbf{B}_{K_\infty}^{[u,v/p]}(j-1),
\end{tikzcd}$$

which is the Koszul complex representing the Galois cohomology of $\mathbf{Q}_p(j-1)$ shifted by $[-1]$. It is clear that the kernel of this map is F^1/F^0, as wanted.

(iii) *Trace maps.* The (φ,Γ)-module trace $\mathrm{Tr}_{\mathrm{Kos}}$ is defined as the "Galois" cohomology of the (φ,Γ)-module residue map followed by $\frac{1}{|\Delta_K|}\mathrm{res}_\pi \circ \mathrm{Tr}_{K_\infty/F_\infty}$. On the other hand, the pro-étale trace is defined as the Galois cohomology of the pro-étale residue map followed by Galois cohomology trace. By point (ii) the first maps of the compositions agree. Hence it suffices to show the compatibility of $\frac{1}{|\Delta_K|}\mathrm{res}_\pi \circ \mathrm{Tr}_{K_\infty/F_\infty}$ with the Galois trace. But this was done in Proposition 3.11. $\square$

[24]In which L has the same meaning as in Section 3.2.1.

7.3.5. *Identification of graded pieces.* Via the comparison morphism α_j from (7.17), and using Lemma 7.19, we get isomorphisms:

$$\begin{aligned}
(7.21)\qquad & \alpha_j^3 : H^i(F^2/F^1)\mathrm{Kos}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}(j)) \xrightarrow{\sim} H^{i-1}(\mathscr{G}_K, \mathbf{Q}_p(j-1)),\\
& \alpha_j^2 : H^i(F^1/F^0)\mathrm{Kos}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}(j)) \xrightarrow{\sim} H^{i-1}(\mathscr{G}_K, \tfrac{\mathscr{O}(Y_C)}{C}(j-1)),\\
& \alpha_j^1 : H^i(F^0/F^{-1})\mathrm{Kos}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}(j)) \xrightarrow{\sim} H^i(\mathscr{G}_K, \mathbf{Q}_p(j)).
\end{aligned}$$

Recall that $(F^1/F^0)\mathrm{Kos}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}(j)) \simeq H^i\mathrm{Kos}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}_0^{[u,v]}(j))$

Lemma 7.22. *We have isomorphisms*

$$H^i\mathrm{Kos}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}_0^{[u,v]}(j)) \xleftarrow{\sim} \begin{cases} \mathscr{O}(Y_K)_0 & \text{if } j=1 \text{ and } i=1,2,\\ 0 & \text{if } j\neq 1 \text{ or } i\neq 1,2.\end{cases}$$

Proof. We have (in $\mathscr{D}(\mathbf{Q}_{p,\square})$)

$$\mathrm{Kos}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}_0^{[u,v]}(j)) \simeq \big[\ \mathrm{Kos}_{\partial,\varphi}(\widetilde{\mathscr{O}}_0^{[u,v]}(j)) \xrightarrow{\gamma_K-1} \mathrm{Kos}_{\partial,\varphi}(\widetilde{\mathscr{O}}_0^{[u,v]}(j))\ \big]$$

and since $t\partial : \widetilde{\mathscr{O}}_0^{[u,v/p]} \to \widetilde{\mathscr{O}}_0^{[u,v/p]}$ is an isomorphism, we get a quasi-isomorphism in $\mathscr{D}(\mathbf{Q}_{p,\square})$:

$$\mathrm{Kos}_{\partial,\varphi}(\widetilde{\mathscr{O}}_0^{[u,v]}(j)) \simeq \big(\ \widetilde{\mathscr{O}}_0^{[u,v]}(j) \xrightarrow{t\partial} \widetilde{\mathscr{O}}_0^{[u,v]}(j-1)\ \big).$$

(The twist is $(j-1)$ and not (j) because of the $\chi(\gamma_K)^{-1}$ appearing in the vertical arrow, necessary to have a commutative diagram as $\gamma_K\cdot t\partial = \chi(\gamma_K)t\partial\cdot\gamma_K$.)

Since $\partial : \widetilde{\mathscr{O}}_0^{[u,v]} \to \widetilde{\mathscr{O}}_0^{[u,v]}$ is an isomorphism, and since $\widetilde{\mathscr{O}}_0^{[u,v]}/t \simeq \mathscr{O}(Y_{\widehat{K}_\infty})_0$, we get an isomorphism

$$(7.23)\qquad H^i\mathrm{Kos}_{\partial,\varphi}(\widetilde{\mathscr{O}}_0^{[u,v]}(j)) \xleftarrow{\sim} \begin{cases} \mathscr{O}(Y_{\widehat{K}_\infty})_0(j-1) & \text{if } i=1,\\ 0 & \text{if } i\neq 1.\end{cases}$$

Finally, we have isomorphisms (the first one is the natural injection, the second one is the cup-product with $\frac{\log\chi}{\log\chi(\gamma_K)}$):

$$H^i(\Gamma, \mathscr{O}(Y_{\widehat{K}_\infty})_0(j-1)) \xleftarrow{\sim} \begin{cases} \mathscr{O}(Y_K)_0 & \text{if } j=1 \text{ and } i=0,1,\\ 0 & \text{if } j\neq 1 \text{ or } i\neq 0,1.\end{cases}$$

This is enough to prove our lemma. ☐

Remark 7.24. (i) In what follows we will use a convenient choice for the isomorphisms in Lemma 7.22 which we will call ω_i, for $i=1,2$, and call ω a lift to the derived category. Set

$$\begin{aligned}
\mathrm{Kos}_\gamma(\mathscr{O}(Y_K)_0) &:= (\mathscr{O}(Y_K)_0 \xrightarrow{\gamma_K-1} \mathscr{O}(Y_K)_0) = (\mathscr{O}(Y_K)_0 \xrightarrow{0} \mathscr{O}(Y_K)_0),\\
\mathrm{Kos}_\gamma(\mathscr{O}(Y_{\widehat{K}_\infty})_0) &:= (\mathscr{O}(Y_{\widehat{K}_\infty})_0 \xrightarrow{\gamma_K-1} \mathscr{O}(Y_{\widehat{K}_\infty})_0).
\end{aligned}$$

We take the following composition (where the second map is induced by the isomorphism $\widetilde{\mathscr{O}}^{[u,v]}/t \simeq \mathscr{O}(Y_{\widehat{K}_\infty})$)

$$\kappa : \mathrm{Kos}_\gamma(\mathscr{O}(Y_K)_0)[-1] \xrightarrow{\text{can}} \mathrm{Kos}_\gamma(\mathscr{O}(Y_{\widehat{K}_\infty})_0)[-1] \longrightarrow \mathrm{Kos}^{\mathrm{HK}}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}_0^{[u,v]}(1)).$$

The isomorphism in Lemma 7.22 comes from the map $\omega := \beta_1^{-1}\kappa$, where β_1 is the map (7.11):

$$\beta_1 : \mathrm{Kos}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}_0^{[u,v]}(1)) \xrightarrow{\sim} \mathrm{Kos}^{\mathrm{HK}}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}_0^{[u,v]}(1)).$$

(ii) Similarly, the isomorphism from (7.23) can be lifted to the derived category. We define it as $\omega_\infty := \beta_{1,\infty}^{-1}\kappa_\infty$, where

$$\begin{aligned}
\kappa_\infty &: \mathscr{O}(Y_{\widehat{K}_\infty})_0[-1] \longrightarrow \mathrm{Kos}^{\mathrm{HK}}_{\partial,\varphi}(\widetilde{\mathscr{O}}_0^{[u,v]}(1)),\\
\beta_{1,\infty} &: \mathrm{Kos}_{\partial,\varphi}(\widetilde{\mathscr{O}}_0^{[u,v]}(1)) \xrightarrow{\sim} \mathrm{Kos}^{\mathrm{HK}}_{\partial,\varphi}(\widetilde{\mathscr{O}}_0^{[u,v]}(1)),
\end{aligned}$$

and the second map is the restriction of β_1.

From the above, we can deduce the following result (compare with Lemma 4.38; of course the splittings depend on the choice of T):

Proposition 7.25. *Let $j \in \mathbf{Z}$. We have $H^i(Y, \mathbf{Q}_p(j)) = 0$ if $i \geq 4$ and, if $i \leq 3$, then we have isomorphisms*

$$H^i(Y, \mathbf{Q}_p(j)) \simeq \begin{cases} \mathscr{O}(Y)_0 \oplus H^i(\mathscr{G}_K, \mathbf{Q}_p(j)) \oplus H^{i-1}(\mathscr{G}_K, \mathbf{Q}_p(j-1)) & \text{if } j = 1 \text{ and } i = 1, 2, \\ H^i(\mathscr{G}_K, \mathbf{Q}_p(j)) \oplus H^{i-1}(\mathscr{G}_K, \mathbf{Q}_p(j-1)) & \text{if } j \neq 1 \text{ or } i \neq 1, 2. \end{cases}$$

7.4. **Identification of the cup-product.**

7.4.1. *Formula for the cup-product.* We are going to use a "basis" e_1, e_2, e_3 to write the differentials in the Koszul complex $\mathrm{Kos}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}(j))$ in a way reminiscent to differentials on differential forms (i.e, e_i behaves as dx_i in computations; we don't explicit the powers of $\chi(\gamma_K)$ involved in the different Tate twists)

$$\begin{aligned} dx &= (\varphi - 1)x \cdot e_1 + (\gamma_K - 1)x \cdot e_2 + t\partial x \cdot e_3, \\ d(ae_1 + be_2 + ce_3) &= (t\partial b - (\gamma_K - 1)c) \cdot e_2 \wedge e_3 + ((\varphi - 1)c - t\partial a) \cdot e_1 \wedge e_3 \\ &\quad + ((\varphi - 1)b - (\gamma_K - 1)a) \cdot e_1 \wedge e_2, \\ d(a \cdot e_2 \wedge e_3 + b \cdot e_1 \wedge e_3 + c \cdot e_1 \wedge e_2) &= ((\varphi - 1)a + (\gamma_K - 1)b - t\partial c) \cdot e_1 \wedge e_2 \wedge e_3. \end{aligned}$$

The cup-product (we use, twice, the formulas from Section 7.3.2 with $\alpha = 1$)

$$\cup^{\mathrm{Kos}} : \mathrm{Kos}^1_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}(j)) \otimes^{\square}_{\mathbf{Q}_p} \mathrm{Kos}^2_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}(2-j)) \to \mathrm{Kos}^3_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}(2))$$

between the terms of degree 1 and 2 is then given by:

$$(a \cdot e_1 + b \cdot e_2 + c \cdot e_3) \cup^{\mathrm{Kos}} (a' \cdot e_2 \wedge e_3 + b' \cdot e_1 \wedge e_3 + c' \cdot e_1 \wedge e_2) = (-a \cup \varphi a' + b \cup \gamma_K b' + c \cup c') \cdot e_1 \wedge e_2 \wedge e_3. \tag{7.26}$$

7.4.2. *Orthogonality and reduction to the Poitou-Tate pairing.* Consider now the trace map

$$\mathrm{Tr}_Y : H^3(\mathrm{Kos}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}(2))) \to \mathbf{Q}_p.$$

Then $\mathrm{Tr}_Y(x \cup^{\mathrm{Kos}} y)$ can be computed in the following way:

- write $x \cup^{\mathrm{Kos}} y$ as $a \cdot e_1 \wedge e_2 \wedge e_3$, with $a \in \widetilde{\mathscr{O}}^{[u,v/p]}$;
- consider the constant term $a_0 \in \mathbf{B}^{[u,v/p]}_{K_\infty}$ of a;
- we have $\mathrm{Tr}_Y(x \cup^{\mathrm{Kos}} y) = \mathrm{Tr}_K \circ h^2_K(a_0)$, where Tr_K is the trace map defined in Section 3.3.

The restriction of the pairing $(x, y) \mapsto \mathrm{Tr}_Y(x \cup^{\mathrm{Kos}} y)$ to 1- and 2-cocycles, respectively, factors through $H^1(Y_K, \mathbf{Q}_p(j)) \otimes^{\square}_{\mathbf{Q}_p} H^2(Y_K, \mathbf{Q}_p(2-j))$ and gives the pairing from Theorem 7.1. Since only the constant term plays a role in the computation of the trace map, we have the following orthogonalities:

$$\begin{aligned} \mathrm{Kos}^1_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}_0(j)) &\perp \big(\mathrm{Kos}^2_{\varphi,\gamma}((\mathbf{B}^{[u,v]}_{K_\infty}(2-j)) \oplus \mathrm{Kos}^1_{\varphi,\gamma}((\mathbf{B}^{[u,v]}_{K_\infty}(1-j))\big), \\ \big(\mathrm{Kos}^1_{\varphi,\gamma}((\mathbf{B}^{[u,v]}_{K_\infty}(j)) \oplus \mathrm{Kos}^0_{\varphi,\gamma}((\mathbf{B}^{[u,v]}_{K_\infty}(j-1))\big) &\perp \mathrm{Kos}^2_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}_0(2-j)) \end{aligned}$$

(because the product of a series with constant term 0 by a constant gives a series with a constant term 0). We also have an orthogonality:

$$\mathrm{Kos}^1_{\varphi,\gamma}((\mathbf{B}^{[u,v]}_{K_\infty}(j)) \perp \mathrm{Kos}^2_{\varphi,\gamma}((\mathbf{B}^{[u,v]}_{K_\infty}(2-j))$$

because all the terms $a \cup \varphi a'$, $b \cup \chi(\gamma_K)^{-1}\gamma_K b'$, $c \cup c'$ are 0. This proves claims (i) of Theorem 7.4 (because the only statement that needs to be checked is that $F^0H^1(Y_K, \mathbf{Q}_p(j)) \otimes^{\square}_{\mathbf{Q}_p} F^0H^3(Y_K, \mathbf{Q}_p(j))$ maps to $F^1H^3(Y_K, \mathbf{Q}_p(j))$. But, in fact, we have just shown that this pairing is 0).

Finally, the restriction to the cocycles in

$$\begin{gathered}\mathrm{Kos}^1_{\varphi,\gamma}((\mathbf{B}^{[u,v]}_{K_\infty}(j))\times \mathrm{Kos}^1_{\varphi,\gamma}((\mathbf{B}^{[u,v]}_{K_\infty}(1-j)),\\ \mathrm{Kos}^0_{\varphi,\gamma}((\mathbf{B}^{[u,v]}_{K_\infty}(j-1))\times \mathrm{Kos}^1_{\varphi,\gamma}((\mathbf{B}^{[u,v]}_{K_\infty}(1-j)),\\ \mathrm{Kos}^0_{\varphi,\gamma}((\mathbf{B}^{[u,v]}_{K_\infty}(j-1))\times \mathrm{Kos}^2_{\varphi,\gamma}((\mathbf{B}^{[u,v]}_{K_\infty}(2-j))\end{gathered}$$

is the usual cup-product coming from the theory of (φ,Γ)-modules and hence is equal to the one from the Poitou-Tate duality. This proves the Galois part of claim (ii) from Theorem 7.4.

7.4.3. *The perfection of the coherent part of the pairing.* It remains to understand the restriction of the pairing to the cocycles in

$$\mathrm{Kos}^1_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}_0(j))\times \mathrm{Kos}^2_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}_0(2-j)). \tag{7.27}$$

If $j\neq 1$, the cohomology groups of both terms in (7.27) are 0 and so the pairing is identically 0. We can thus assume that $j=1$ and in that case we have, by Lemma 7.22, the isomorphisms:

$$\omega_1:\mathscr{O}(Y)_0\xrightarrow{\sim} H^1(\mathrm{Kos}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}_0(1))),\quad \omega_2:\mathscr{O}(Y)_0\xrightarrow{\sim} H^2(\mathrm{Kos}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}_0(1))).$$

Lemma 7.28. *The following diagram commutes:*

$$\begin{array}{ccc}\mathscr{O}(Y)_0\otimes^\square_{\mathbf{Q}_p}\mathscr{O}(Y)_0 & \xrightarrow{\ \cup^{\mathrm{coh}}\ } & \mathbf{Q}_p\\ \wr\downarrow\omega_1 \qquad \wr\downarrow\omega_2 & & \|\\ H^1(\mathrm{Kos}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}_0(1)))\otimes^\square_{\mathbf{Q}_p} H^2(\mathrm{Kos}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}_0(1))) & \xrightarrow{\mathrm{Tr}_Y\circ\cup^{\mathrm{Kos}}} & \mathbf{Q}_p\end{array}$$

In particular, since the top pairing is perfect so is the bottom pairing.

Proof. We start with $f,g\in\mathscr{O}(Y)_0$, and we will construct cocycles

$$z^1(g)\in \mathrm{Kos}^1_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}_0(1)),\quad z^2(f)\in \mathrm{Kos}^2_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}_0(1))$$

representing $\omega_1(g)$ and $\omega_2(f)$. Then we are going to show that:

$$\mathrm{Tr}_Y\big(\mathrm{cl}(z^1(g))\cup^{\mathrm{Kos}}\mathrm{cl}(z^2(f))\big)=\tfrac{1}{\log\chi(\gamma_K)}\mathrm{Tr}_{K/\mathbf{Q}_p}(g\cup^{\mathrm{coh}}f). \tag{7.29}$$

• For g, take its image $\kappa(g)\in \mathrm{Kos}^{\mathrm{HK},1}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}_0(1))$ (see remark 7.24 for the definition of the map κ). Now, take $\tilde g\in\widetilde{\mathscr{O}}^{[u,v]}$ that lifts g and consider the cocycle (in $\mathrm{Kos}^1_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}_0(1))$)

$$z^1(g):=-((\varphi-1)\tfrac{\tilde g}{t}(1)\cdot e_1+(\gamma_K-1)\tfrac{\tilde g}{t}(1)\cdot e_2+t\partial\tfrac{\tilde g}{t}\cdot e_3).$$

(The twist (1) plays a role only in the action of γ_K and compensates for the action on t; it follows that $(\gamma_K-1)\frac{\tilde g}{t}(1)\in\widetilde{\mathscr{O}}^{[u,v]}$ since g is fixed by γ_K. The cocycle $z^1(g)$ has then values in the desired group and is not a coboundary as $\frac{\tilde g}{t}\notin\widetilde{\mathscr{O}}^{[u,v]}$ when $g\neq 0$.) We easily check that $\mathrm{cl}(\beta_1(z^1(g)))=\mathrm{cl}(\kappa(g))$ (see formula (7.11) for the definition of the map β_1); hence $\omega_1(g)$ is represented by $z^1(g)$.

• For f, take its image $\kappa(f)\in\mathrm{Kos}^{\mathrm{HK},2}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}_0(1))$. Now, take $\tilde f\in\widetilde{\mathscr{O}}^{[u,v]}$ that lifts f and consider the cocycle (in $\mathrm{Kos}^2_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}_0(1))$)

$$z^2(f):=-\partial\tilde f\cdot e_2\wedge e_3-(\varphi-1)\tfrac{\tilde f}{t}(1)\cdot e_1\wedge e_2.$$

We easily check that $\mathrm{cl}(\beta_1(z^2(f)))=\mathrm{cl}(\kappa(f))$; hence $\omega_2(f)$ is represented by $z^2(f)$.

• Then (use formula (7.26))

$$z^1(g)\cup^{\mathrm{Kos}}z^2(f)=[-(\varphi-1)\tfrac{\tilde g}{t}\cdot\varphi(\partial\tilde f)+t\partial\tfrac{\tilde g}{t}\cdot(\varphi-1)\tfrac{\tilde f}{t}]e_1\wedge e_2\wedge e_3.$$

We can write $\tilde f$ and $\tilde g$ as series in T and reduce to the case $\tilde f=\alpha T^i$ and $\tilde g=\beta T^j$, for $i,j\neq 0$, (and so $f=\theta(\alpha)T^i$, $g=\theta(\beta)T^j$). Using $\varphi(T)=T^p$, $\varphi(t)=pt$, $\partial T^k=kT^k$, we obtain the formula

$$z^1(g)\cup^{\mathrm{Kos}}z^2(f)=-\big[\big(\tfrac{\varphi(\beta)T^{pj}}{pt}-\tfrac{\beta T^j}{t}\big)i\varphi(\alpha)T^{pi}-j\beta T^j\big(\tfrac{\varphi(\alpha)T^{pi}}{pt}-\tfrac{\alpha T^i}{t}\big)\big]e_1\wedge e_2\wedge e_3.$$

In order to get a nonzero constant term we need $j+pi=0$ or $i+j=0$.

— If $j+pi=0$, the constant term is $-(-\beta i\varphi(\alpha)-\frac{j\beta\varphi(\alpha)}{p})\frac{1}{t}=0$.

— If $i+j=0$, the constant term is $-\frac{i\varphi(\alpha)\varphi(\beta)}{p\,t}-\frac{j\alpha\beta}{t}=j(\varphi-1)\frac{\alpha\beta}{t}$.

It follows from Proposition 3.12 that

$$\operatorname{Tr}_Y(z^1(g)\cup^{\mathrm{Kos}} z^2(f))=\begin{cases}\frac{-[K:\mathbf{Q}_p]}{\log\chi(\gamma_K)}\,j\operatorname{Tr}(\theta(\alpha)\theta(\beta)) & \text{if } i+j=0,\\ 0 & \text{if } i+j\neq 0,\end{cases}$$

Using that $\operatorname{Tr}_{K/\mathbf{Q}_p}=[K:\mathbf{Q}_p]\operatorname{Tr}$ on K, we deduce that (see (6.6))

$$\operatorname{Tr}_Y(z^1(g)\cup^{\mathrm{Kos}} z^2(f))=\tfrac{-1}{\log\chi(\gamma_K)}\operatorname{Tr}_{K/\mathbf{Q}_p}(\operatorname{res}(fdg))=\tfrac{1}{\log\chi(\gamma_K)}\operatorname{Tr}_{K/\mathbf{Q}_p}(g\cup^{\mathrm{coh}} f).$$

This proves equality (7.29), which we wanted. □

We have proved that the pairing

$$\operatorname{Tr}_Y\circ\cup^{\mathrm{Kos}}: H^1((F^1/F^0)\mathrm{Kos}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}(j)))\otimes^{\square}_{\mathbf{Q}_p} H^2((F^1/F^0)\mathrm{Kos}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}^{[u,v]}(2-j)))\to\mathbf{Q}_p$$

is perfect since or it is trivial or a multiple (for $j=1$) of the coherent pairing

$$\operatorname{Tr}_{K/\mathbf{Q}_p}\circ\cup^{\mathrm{coh}}:\frac{\mathscr{O}(Y_K)}{K}\otimes^{\square}_{\mathbf{Q}_p}\frac{\mathscr{O}(Y_K)}{K}\to\mathbf{Q}_p.$$

It follows, by (7.21), that the pairing

$$\cup^{\text{proét}}: H^0(\mathscr{G}_K,(\tfrac{\mathscr{O}(Y_C)}{C})(j-1))\otimes^{\square}_{\mathbf{Q}_p} H^1(\mathscr{G}_K,(\tfrac{\mathscr{O}(Y_C)}{C})(1-j))\to\mathbf{Q}_p \tag{7.30}$$

induced from pro-étale pairing is also perfect.

7.4.4. *Identification of the pairing* (7.30). To finish the proof of Theorem 7.4, it remains to show that, for $j=1$, the pairing (7.30) is equal to the one induced from Galois pairing and coherent pairing:

$$\cup^{\mathrm{Gal}}: H^0(\mathscr{G}_K,\tfrac{\mathscr{O}(Y_C)}{C})\otimes^{\square}_{\mathbf{Q}_p} H^1(\mathscr{G}_K,\tfrac{\mathscr{O}(Y_C)}{C})\xrightarrow{\cup} H^1(\mathscr{G}_K,C)\xleftarrow[\sim]{\frac{\log\chi}{\log\chi(\gamma_K)}} K\xrightarrow{\operatorname{Tr}_K}\mathbf{Q}_p.$$

But, since both Galois cohomology groups $H^0(\mathscr{G}_K,\frac{\mathscr{O}(Y_C)}{C})$ and $H^1(\mathscr{G}_K,\frac{\mathscr{O}(Y_C)}{C})$ are isomorphic to $\frac{\mathscr{O}(Y_K)}{K}$, this can be checked by pulling back these pairings to $\frac{\mathscr{O}(Y_K)}{K}$. By Proposition 7.32 below, the pro-étale pairing pullbacks to the coherent pairing $\cup^{\mathrm{coh}}$; by Lemma 7.31 below, so does the Galois-coherent pairing $\cup^{\mathrm{Gal}}$. This proves that $\cup^{\text{proét}}=\cup^{\mathrm{Gal}}$, as wanted.

Lemma 7.31. *The following diagram is commutative:*

$$\begin{tikzcd}
H^0(\mathscr{G}_K,\frac{\mathscr{O}(Y_C)}{C})\otimes^{\square}_{\mathbf{Q}_p} H^1(\mathscr{G}_K,\frac{\mathscr{O}(Y_C)}{C}) \arrow[r, "\cup"] & H^1(\mathscr{G}_K,C) & K \arrow[l, "\frac{\log\chi}{\log\chi(\gamma_K)}"', "\sim"] \arrow[r, "\operatorname{Tr}_K"] \arrow[d, equal] & \mathbf{Q}_p \arrow[d, equal] \\
\frac{\mathscr{O}(Y)}{K}\otimes^{\square}_{\mathbf{Q}_p}\frac{\mathscr{O}(Y)}{K} \arrow[u, "\wr\ \mathrm{can}\ \otimes\ \wr\ \cup\frac{\log\chi}{\log\chi(\gamma_K)}"'] \arrow[rr, "\cup"] & & K \arrow[r, "\operatorname{Tr}_K"] & \mathbf{Q}_p.
\end{tikzcd}$$

In particular, the Galois-coherent pairing defined by the top row is perfect.

Proof. We compute. Going up and then right in the above diagram (and stopping at K) we get:

$$\begin{aligned}\{f,g\}\to\{f,\{\sigma\mapsto\tfrac{\log\chi(\sigma)}{\log\chi(\gamma_K)}g\}\}&\xrightarrow{\cup}\{\sigma\mapsto\operatorname{res}(fd(\tfrac{\log\chi(\sigma)}{\log\chi(\gamma_K)}g))\}\\ &=\{\sigma\mapsto\tfrac{\log\chi(\sigma)}{\log\chi(\gamma_K)}\operatorname{res}(fdg)\}.\end{aligned}$$

Going right and then up we get:

$$\{f,g\}\xrightarrow{\cup}\operatorname{res}(fdg)\to\{\sigma\mapsto\tfrac{\log\chi(\sigma)}{\log\chi(\gamma_K)}\operatorname{res}(fdg)\},$$

as wanted. □

Proposition 7.32. *Let $i = 1, 2$. The following diagram commutes*

$$\begin{tikzcd} H^i\mathrm{Kos}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}_0^{[u,v]}(1)) \arrow[r, "\alpha_1^2", "\sim"'] & H^{i-1}(\mathscr{G}_K, \frac{\mathscr{O}(Y_C)}{C}) \\ \mathscr{O}(\mathrm{Y}_K)/K, \arrow[u, "\omega_i", "\wr"'] \arrow[ur, "f_i", "\sim"'] & \end{tikzcd}$$

where $f_1 = \mathrm{can}$, $f_2 = \frac{\log\chi}{\log\chi(\gamma_K)}$.

Proof. Consider the commutative diagram (we shortened Kos to K, $\mathrm{res}_{\mathrm{proét}}$ to res; removed subscripts from pro-étale cohomology; and set $s := i-1$):

$$\begin{tikzcd}
H^s\mathrm{K}_\gamma(\mathscr{O}(Y_K)_0) \arrow[r, "\sim"] \arrow[d, "\mathrm{can}", "\wr"'] \arrow[ddd, bend right=60, "\omega_i"'] & \mathscr{O}(Y_K)_0 \arrow[d, "\wr"] \arrow[ddddd, bend left=80, "f_i"] & \\
H^s\mathrm{K}_\gamma(\mathscr{O}(Y_{\widehat{K}_\infty})_0) \arrow[r, equal] \arrow[d, "\kappa", "\wr"'] & H^s\mathrm{K}_\gamma(\mathscr{O}(Y_{\widehat{K}_\infty})_0) \arrow[d, "\kappa_\infty", "\wr"'] & H^s(\Gamma, \mathscr{O}(Y_{\widehat{K}_\infty})_0) \arrow[l, "\gamma_K"', "\sim"] \arrow[lddd, bend left=60, "\mu_\infty", "\wr"'] \\
H^{s+1}\mathrm{K}^{\mathrm{HK}}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}_0^{[u,v]}(1))) \arrow[r] & H^s\mathrm{K}_{\gamma_K}(H^1\mathrm{K}^{\mathrm{HK}}_{\partial,\varphi}(\widetilde{\mathscr{O}}_0^{[u,v]}(1))) & \\
H^{s+1}\mathrm{K}_{\partial,\varphi,\gamma}(\widetilde{\mathscr{O}}_0^{[u,v]}(1))) \arrow[r] \arrow[u, "\beta_1", "\wr"'] \arrow[d, "\alpha_1"] & H^s\mathrm{K}_\gamma(H^1\mathrm{K}_{\partial,\varphi}(\widetilde{\mathscr{O}}_0^{[u,v]}(1))) \arrow[u, "\beta_{1,\infty}", "\wr"'] \arrow[d, "\alpha_{1,\infty}"] & \\
H^{s+1}(Y, \mathbf{Q}_p(1))^{\mathrm{res}=0} \arrow[r] \arrow[dr, bend right, "\alpha_1^2"'] & H^s(\mathscr{G}_K, H^1(Y_C, \mathbf{Q}_p(1))^{\mathrm{res}=0}) & \\
& H^s(\mathscr{G}_K, \mathscr{O}(Y_C)_0) \arrow[u, "\wr"] &
\end{tikzcd}$$

The bottom square commutes by the proof of Lemma 7.19. The rest commutes basically by constructions of the involved maps. Tracing this diagram proves our proposition. □

8. Arithmetic Poincaré duality

Let K be a finite extension of $\mathbf{Q}_p$. This chapter is devoted to the proof of the arithmetic Poincaré duality for smooth dagger curves over K. We start with stating the duality, then we prove it for proper curves, where it is an easy consequence of the geometric Poincaré duality. After that we prove it for an open disc and an open annulus over K (via a reduction to the Poincaré duality for the ghost circle proved earlier). This then allows us to treat the case of wide open curves (a special type of Stein curves) and we treat the case of general Stein curves by a limit argument. Finally, an analogous limit argument yields the duality for a dagger affinoid.

8.1. **The statement of arithmetic Poincaré duality.** The goal of this paper is to prove the following theorem:

Theorem 8.1. (Arithmetic Poincaré duality) *Let X be a smooth geometrically irreducible dagger variety of dimension* 1 *over K. Assume that X is proper, Stein, or affinoid. Then:*

(1) There is a natural trace map isomorphism

$$\mathrm{Tr}_X : H^4_c(X, \mathbf{Q}_p(2)) \xrightarrow{\sim} \mathbf{Q}_p.$$

(2) For $i, j \in \mathbf{Z}$, the pairing

$$H^i(X, \mathbf{Q}_p(j)) \otimes^{\square}_{\mathbf{Q}_p} H^{4-i}_c(X, \mathbf{Q}_p(2-j)) \xrightarrow{\sim} H^4_c(X, \mathbf{Q}_p(2)) \xrightarrow[\sim]{\mathrm{Tr}_X} \mathbf{Q}_p$$

is a perfect duality, i.e., we have induced isomorphisms

$$\gamma_{X,i} : H^i(X, \mathbf{Q}_p(j)) \xrightarrow{\sim} H^{4-i}_c(X, \mathbf{Q}_p(2-j))^*,$$
$$\gamma^c_{X,i} : H^i_c(X, \mathbf{Q}_p(j)) \xrightarrow{\sim} H^{4-i}(X, \mathbf{Q}_p(2-j))^*.$$

Here we wrote $(-)^*$ for the internal Hom in the category of solid $\mathbf{Q}_p$-vector spaces.

8.2. **The case of proper curves.** A proper smooth curve is the analytification of an algebraic smooth curve for which Theorem 8.1 has been known for quite a while[25]. But, actually, Theorem 8.1 holds for any smooth proper geometrically irreducible rigid analytic variety X over K of dimension d (see Corollary 8.3). This follows from the recently proved geometric Poincaré duality (Theorem 8.2) and local Galois duality.

Recall that, if X is a smooth proper variety over K, geometrically irreducible, then its pro-étale cohomology complex $\mathrm{R}\Gamma(X_L, \mathbf{Q}_p(j))$, for $L = K, C$, has classical cohomology and the cohomology groups $H^i(X_L, \mathbf{Q}_p(j))$ are finite dimensional $\mathbf{Q}_p$-vector spaces with their canonical Hausdorff topology (see Section 5.1.1). Over C, it satisfies Poincaré duality:

Theorem 8.2. (Zavyalov, Mann [34, 5.5.7], [25, Th. 1.1.1]) *Let X be a smooth proper geometrically irreducible rigid analytic variety of pure dimension d over K. Then:*

(i) There is a Galois-equivariant $\mathbf{Q}_p$-linear trace map isomorphism

$$\mathrm{Tr}_{X_C} : H^{2d}(X_C, \mathbf{Q}_p(d)) \xrightarrow{\sim} \mathbf{Q}_p.$$

(ii) For $i \in \mathbf{N}, j \in \mathbf{Z}$, the trace map Tr_{X_C} induces a perfect Galois-equivariant pairing of finite rank $\mathbf{Q}_p$-vector spaces:

$$H^i(X_C, \mathbf{Q}_p(j)) \otimes^{\square}_{\mathbf{Q}_p} H^{2d-i}(X_C, \mathbf{Q}_p(d-j)) \xrightarrow{\sim} H^{2d}(X_C, \mathbf{Q}_p(d)) \xrightarrow[\sim]{\mathrm{Tr}_{X_C}} \mathbf{Q}_p.$$

Combining it with local Galois duality we obtain:

Corollary 8.3. (*Arithmetic Poincaré duality*) *Let X be a smooth proper geometrically irreducible rigid analytic variety of pure dimension d over K. Then:*

(i) There is a natural $\mathbf{Q}_p$-linear trace map isomorphism

$$\mathrm{Tr}_X : H^{2d+2}(X, \mathbf{Q}_p(d+1)) \xrightarrow{\sim} \mathbf{Q}_p.$$

(ii) For $i \in \mathbf{N}, j \in \mathbf{Z}$, the pairing

$$H^i(X, \mathbf{Q}_p(j)) \otimes^{\square}_{\mathbf{Q}_p} H^{2d+2-i}(X, \mathbf{Q}_p(d+1-j)) \xrightarrow{\sim} H^{2d+2}(X, \mathbf{Q}_p(d+1)) \xrightarrow[\sim]{\mathrm{Tr}_X} \mathbf{Q}_p$$

is a perfect duality of finite rank $\mathbf{Q}_p$-vector spaces.

Proof. This follows from Theorem 8.2 and from the Hochschild-Serre spectral sequence:

$$E_2^{a,b}(j) = H^a(\mathscr{G}_K, H^b(X_C, \mathbf{Q}_p(j))) \Rightarrow H^{a+b}(X, \mathbf{Q}_p(j)). \tag{8.4}$$

The trace map Tr_X comes from the composition:

$$\mathrm{Tr}_X : H^{2d+2}(X, \mathbf{Q}_p(d+1)) \simeq H^2(\mathscr{G}_K, H^{2d}(X_C, \mathbf{Q}_p(d+1))) \xrightarrow[\sim]{\mathrm{Tr}_{X_C}(1)} H^2(\mathscr{G}_K, \mathbf{Q}_p(1)) \xrightarrow{\sim} \mathbf{Q}_p.$$

By Theorem 8.2, it is an isomorphism.

To prove the duality, note that the only nonzero terms of $E_2^{a,b}(j)$ are those with degrees $0 \leq a \leq 2$ and $0 \leq b \leq 2d$; hence the spectral sequence degenerates at E_3. As the cup product commutes

[25] Of course, modulo certain identifications.

with the differentials $d_2^{a,b}: E_2^{a,b} \to E_2^{a+2,b-1}$ of (8.4), we get a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & E_3^{a,b}(j) \arrow[r] \arrow[d] & H^a(\mathscr{G}_K, H^b(j)) \arrow[r, "d"] \arrow[d, "\wr"] & H^{a+2}(\mathscr{G}_K, H^{b-1}(j)) \arrow[r] \arrow[d, "\wr"] & E_3^{a+2,b-1}(j) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E_3^{2-a,2d-b}(j^*)^* \arrow[r] & H^{2-a}(\mathscr{G}_K, H^{2d-b}(j^*))^* \arrow[r, "\tilde{d}^*"] & H^{-a}(\mathscr{G}_K, H^{2d+1-b}(j^*))^* \arrow[r] & E_3^{-a,2d+1-b}(j^*)^* \arrow[r] & 0
\end{tikzcd}$$

where $d := d_2^{a,b}$, $\tilde{d} := d_2^{-a,2d+1-b}$ and we set $j^* := d+1-j$ and $H^b(j) := H^b(X_C, \mathbf{Q}_p(j))$. The second and third vertical arrows are isomorphisms by Theorem 8.2 and Tate's duality. We deduce that the two other vertical maps are isomorphisms as well.

The Hochschild-Serre spectral sequence (8.4) induces a descending filtration on $H^i(X, \mathbf{Q}_p(j))$:

$$H^i(X, \mathbf{Q}_p(j)) = F^0_{i,j} \supset F^1_{i,j} \supset F^2_{i,j} \supset F^3_{i,j} = 0.$$

Since it degenerates at E_3, i.e., $E_3 = E_\infty$, the above diagram gives a perfect duality

$$\mathrm{gr}_F^{\bullet} H^i(X, \mathbf{Q}_p(j)) \otimes^{\square}_{\mathbf{Q}_p} \mathrm{gr}_F^{2-\bullet} H^{2d+2-i}(X, \mathbf{Q}_p(d+1-j)) \to \mathrm{gr}_F^2 H^{2d+2}(X, \mathbf{Q}_p(d+1)) \xrightarrow{\sim} \mathbf{Q}_p.$$

By devissage (via the filtration F), this perfect duality lifts to the following perfect pairings (in the presented order):

$$(F^0/F^2)H^i(X, \mathbf{Q}_p(j)) \otimes^{\square}_{\mathbf{Q}_p} F^1 H^{2d+2-i}(X, \mathbf{Q}_p(d+1-j)) \to \mathbf{Q}_p,$$
$$F^0 H^i(X, \mathbf{Q}_p(j)) \otimes^{\square}_{\mathbf{Q}_p} F^0 H^{2d+2-i}(X, \mathbf{Q}_p(d+1-j)) \to \mathbf{Q}_p.$$

This concludes the proof. $\square$

We specialize now to curves. Let X be a proper smooth geometrically irreducible curve over K. By [1, Rem. 7.16], our geometric trace map $\mathrm{Tr}_{X_C}: H^2(X_C, \mathbf{Q}_p(1)) \to \mathbf{Q}_p$ is equal to the trace map used by Zavyalov in [34], i.e., to the rigid analytic version of the Berkovich trace map (see [34, Sec. 5.3], [4, Sec. 7.2]). Hence in this case Theorem 8.1 follows from Zavyalov's arithmetic duality stated in Corollary 8.3.

8.3. **The case of an open disc.** We will now prove Theorem 8.1 for an open disc. Let D be an open disc D over K.

Proposition 8.5. (Arithmetic duality for an open disc) *Theorem 8.1 holds for D.*

Proof. (i) The trace map

$$\mathrm{Tr}_D: H^4_c(D, \mathbf{Q}_p(2)) \to \mathbf{Q}_p$$

was defined in Section 6.1. It will be convenient to have a different description of this map. Let $Y := \partial D$ be the boundary of D, a ghost circle. Consider the composition

$$t_D: H^4_c(D, \mathbf{Q}_p(2)) \underset{\sim}{\overset{\partial}{\leftarrow}} H^3(Y, \mathbf{Q}_p(2)) \xrightarrow[\sim]{\mathrm{Tr}_Y} \mathbf{Q}_p.$$

The first strict isomorphism holds because $H^i(D, \mathbf{Q}_p(2)) = 0$, for $i \geq 3$, by Lemma 4.27. An alternative definition of t_D is the following. First, we define the geometric trace zig-zag:

$$t_{D_C}: H^2_c(D_C, \mathbf{Q}_p(1)) \overset{\partial}{\leftarrow} H^1(Y_C, \mathbf{Q}_p(1)) \xrightarrow{\mathrm{Tr}_{Y_C}} \mathbf{Q}_p.$$

Then the arithmetic trace t_D is obtained as a composition of $H^2(\mathscr{G}_K, t_{D_C}(1))$ with $\mathrm{Tr}_K: H^2(\mathscr{G}_K, \mathbf{Q}_p(1)) \xrightarrow{\sim} \mathbf{Q}_p$ via the identifications coming from (5.19) and Lemma 4.38.

We claim that $t_D = \mathrm{Tr}_D$. For that, it is enough to prove that the geometric traces $t_{D_C}, \mathrm{Tr}_{D_C}$ are equal after we apply $H^2(\mathscr{G}_K, (-)(1))$. This will follow if we prove that the following diagram commutes

$$\begin{tikzcd}
H^1(Y_C, \mathbf{Q}_p(1)) \arrow[r, "\mathrm{Tr}_{Y_C}"] \arrow[d, "\partial"] & \mathbf{Q}_p \\
H^2_c(D_C, \mathbf{Q}_p(1)) \arrow[ur, "\mathrm{Tr}_{D_C}"'] &
\end{tikzcd}$$

Or, as can be seen by unwinding the definitions of trace maps, that the following diagram commutes

$$\begin{tikzcd}
H^1_{\mathrm{syn}}(Y_C,2) \arrow[r] \arrow[d, "\partial"] & (H^1_{\mathrm{HK}}(Y_C)\otimes^{\square}_{\breve{F}}\widehat{\mathbf{B}}^+_{\mathrm{st}})^{N=0,\varphi=p^2} \arrow[d, "\partial"] \\
H^2_{\mathrm{syn},c}(D_C,2) \arrow[r] & (H^2_{\mathrm{HK},c}(D_C)\otimes^{\square}_{\breve{F}}\widehat{\mathbf{B}}^+_{\mathrm{st}})^{N=0,\varphi=p^2}
\end{tikzcd}$$

But this follows easily from the definitions.

(ii) We will first show that the pairing

$$H^i(D,\mathbf{Q}_p(j))\otimes^{\square}_{\mathbf{Q}_p} H^{4-i}_c(D,\mathbf{Q}_p(2-j)) \xrightarrow{\cup} H^4_c(D,\mathbf{Q}_p(2)) \xrightarrow[\sim]{\mathrm{Tr}_D} \mathbf{Q}_p \tag{8.6}$$

induces the isomorphism

$$\gamma_{D,i}: \quad H^i(D,\mathbf{Q}_p(j)) \xrightarrow{\sim} H^{4-i}_c(D,\mathbf{Q}_p(2-j))^*. \tag{8.7}$$

(•) *Compatibility of pairings.* The pairing (8.6) is compatible with the pairing (7.2), i.e., the diagram

$$\begin{tikzcd}
H^i(D,\mathbf{Q}_p(j))\otimes^{\square}_{\mathbf{Q}_p} H^{i'}_c(D,\mathbf{Q}_p(j')) \arrow[r, "\cup"] \arrow[d, hook, "\mathrm{can}"] & H^{i+i'}_c(D,\mathbf{Q}_p(j+j')) \\
H^i(Y,\mathbf{Q}_p(j))\otimes^{\square}_{\mathbf{Q}_p} H^{i'-1}(Y,\mathbf{Q}_p(j')) \arrow[r, "\cup"] \arrow[u, two heads, "\partial"] & H^{i+i'-1}(Y,\mathbf{Q}_p(j+j')) \arrow[u, two heads, "\partial"']
\end{tikzcd} \tag{8.8}$$

commutes (up to a sign): for $a\in H^i(D,\mathbf{Q}_p(j))(S)$ and $b\in H^{i'-1}(Y,\mathbf{Q}_p(j'))(S)$, where S is an extremally disconnected set, we have

$$\partial(\mathrm{can}(a)\cup b)=(-1)^i a\cup\partial(b).$$

This follows easily from the formula (7.15) (take $\alpha=0$). The injectivity and surjectivity of the vertical maps in the diagram follow from (5.16).

(•) *Filtration on cohomology.* By Section 4.4 and Lemma 4.27, there exists an ascending filtration on $H^i(D,\mathbf{Q}_p(j))$:

$$F^2_{i,j}=H^i(D,\mathbf{Q}_p(j))\supset F^1_{i,j}\supset F^0_{i,j}\supset F^{-1}_{i,j}=0,$$

such that

$$F^1_{i,j}=F^2_{i,j}=H^i(D,\mathbf{Q}_p(j)),\quad F^1_{i,j}/F^0_{i,j}\simeq H^{i-1}(\mathscr{G}_K,\tfrac{\mathscr{O}(D_C)}{C}(j-1)),$$
$$F^0_{i,j}/F^{-1}_{i,j}\simeq H^i(\mathscr{G}_K,\mathbf{Q}_p(j)).$$

Lemma 8.9. *The injection*

$$\mathrm{can}: H^i(D,\mathbf{Q}_p(j))\hookrightarrow H^i(Y,\mathbf{Q}_p(j))$$

is strict for the given filtrations, i.e., the induced map

$$(F^{s+1}/F^s)H^i(D,\mathbf{Q}_p(j))\to(F^{s+1}/F^s)H^i(Y,\mathbf{Q}_p(j)),\quad s\geq -1, \tag{8.10}$$

is injective.

Proof. Since $(F^{s+1}/F^s)H^i(D,\mathbf{Q}_p(j))=0$ for $s\neq -1,0$, it suffices to check the statement of the lemma for $s=-1,0$. For $s=-1$, it is clear. For $s=0$, we can write the map (8.10) as:

$$H^{i-1}(\mathscr{G}_K,\tfrac{\mathscr{O}(D_C)}{C}(j-1))\longrightarrow H^{i-1}(\mathscr{G}_K,\tfrac{\mathscr{O}(Y_C)}{C}(j-1)). \tag{8.11}$$

Our claim now follows from the compatible isomorphisms (3.16). □

($\bullet$) *Filtration on cohomology with compact support.* There exists an ascending filtration on $H^i_c(D,\mathbf{Q}_p(j))$:

$$F^2_{c,i,j}=H^i_c(D,\mathbf{Q}_p(j))\supset F^1_{c,i,j}\supset F^0_{c,i,j}=0,$$

such that

$$F^2_{c,i,j}/F^1_{c,i,j}\simeq H^{i-2}(\mathscr{G}_K,\mathbf{Q}_p(j-1)),\quad F^1_{c,i,j}/F^0_{c,i,j}\simeq H^{i-2}(\mathscr{G}_K,\tfrac{\mathscr{O}(Y_C)}{\mathscr{O}(D_C)}(j-1)).$$

We can visualize it in the following way (to simplify the notation we removed the subscripts from cohomology):

$$\begin{tikzcd}
 & 0 \arrow[d] \\
F^1_{c,i,j} \arrow[r, "\sim"] \arrow[d] & H^{i-2}(\mathscr{G}_K,(\mathscr{O}(Y_C)/\mathscr{O}(D_C))(j-1)) \arrow[d] \\
F^2_{c,i,j}:=H^i_c(D,\mathbf{Q}_p(j)) \arrow[r, "\sim"] & H^{i-2}(\mathscr{G}_K,H^2_c(D_C,\mathbf{Q}_p(j))) \arrow[d] \\
 & H^{i-2}(\mathscr{G}_K,\mathbf{Q}_p(j-1)) \arrow[d] \\
 & 0
\end{tikzcd}$$

Hence the filtration comes basically only from the syntomic sequence. The right column is exact by Lemma 5.15.

Lemma 8.12. *The canonical maps*

$$\partial: F^sH^{i-1}(Y,\mathbf{Q}_p(j))\to F^sH^i_c(D,\mathbf{Q}_p(j)),\quad s\geq -1,$$

are surjective.

Proof. This is clear for $s=-1,0$ and $s\geq 2$. For $s=1$, we use the computations done earlier, to rewrite the map in the lemma as the canonical map:

$$\partial: H^{2-i}(\mathscr{G}_K,\tfrac{\mathscr{O}(Y_C)}{C}(1-j))\to H^{2-i}(\mathscr{G}_K,\tfrac{\mathscr{O}(Y_C)}{\mathscr{O}(D_C)}(1-j)). \tag{8.13}$$

Using the generalized Tate's isomorphisms (3.16) we see that the map in (8.13) is surjective. $\square$

($\bullet$) *Pairings on the graded pieces.* From diagram (8.8), Lemma 8.12, and Theorem 7.4, it follows that the cup product pairing

$$\cup:\quad H^i(D,\mathbf{Q}_p(j))\otimes^{\square}_{\mathbf{Q}_p}H^{i'}_c(D,\mathbf{Q}_p(j'))\to H^{i+i'}_c(D,\mathbf{Q}_p(j+j'))$$

is compatible with the above filtrations. In particular, the subgroups

$$\begin{aligned}
&F^0_{i,j}=H^i(\mathscr{G}_K,\mathbf{Q}_p(j))\subset H^i(D,\mathbf{Q}_p(j)),\\
&F^1_{c,4-i,2-j}=H^{2-i}(\mathscr{G}_K,\tfrac{\mathscr{O}(Y_C)}{\mathscr{O}(D_C)}(1-j))\subset H^{4-i}_c(D,\mathbf{Q}_p(2-j))
\end{aligned}$$

are orthogonal. That orthogonality induces the following map of exact sequences (all the vertical maps are induced from cup products and the trace map Tr_D):

$$\begin{tikzcd}
0 \arrow[r] & H^i(\mathscr{G}_K,\mathbf{Q}_p(j)) \arrow[r] \arrow[d, "\alpha_{D,i}"] & H^i(D,\mathbf{Q}_p(j)) \arrow[r] \arrow[d, "\gamma_{D,i}"] & H^{i-1}(\mathscr{G}_K,\frac{\mathscr{O}(D_C)}{C}(j-1)) \arrow[r] \arrow[d, "\beta_{D,i}"] & 0 \\
0 \arrow[r] & H^{2-i}(\mathscr{G}_K,\mathbf{Q}_p(1-j))^* \arrow[r] & H^{4-i}_c(D,\mathbf{Q}_p(2-j))^* \arrow[r] & H^{2-i}(\mathscr{G}_K,\frac{\mathscr{O}(Y_C)}{\mathscr{O}(D_C)}(1-j))^* \arrow[r] & 0
\end{tikzcd} \tag{8.14}$$

It suffices now to show that maps $\alpha_{D,i}$ and $\beta_{D,i}$ are isomorphisms. This is easy to prove for the first map because this map is induced from Galois cohomology pairing

$$H^i(\mathscr{G}_K,\mathbf{Q}_p(j))\otimes^{\square}_{\mathbf{Q}_p}H^{2-i}(\mathscr{G}_K,\mathbf{Q}_p(1-j))\xrightarrow{\cup}H^2(\mathscr{G}_K,\mathbf{Q}_p(1))\xrightarrow[\sim]{\mathrm{Tr}_K}\mathbf{Q}_p$$

as can be seen by comparing $\alpha_{D,i}$ with the map $\alpha_{Y,i}$ – an analog for the ghost circle Y – and evoking Theorem 7.4.

To identify map β_D, consider the commutative diagram (we omitted the indices of filtrations and subscripts from cohomology):

$$\begin{tikzcd}
(F^1/F^0)H^i(D,\mathbf{Q}_p(j))\otimes^{\square}_{\mathbf{Q}_p}(F^1_c/F^0_c)H^{4-i}_c(D,\mathbf{Q}_p(2-j)) \arrow[r, "\cup"] \arrow[d, hook, "\mathrm{can}"] & (F^2_c/F^1_c)H^4_c(D,\mathbf{Q}_p(2)) \arrow[r, "\mathrm{Tr}_D", "\sim"'] & \mathbf{Q}_p \\
(F^1/F^0)H^i(Y,\mathbf{Q}_p(j))\otimes^{\square}_{\mathbf{Q}_p}(F^1/F^0)H^{3-i}(Y,\mathbf{Q}_p(2-j)) \arrow[r, "\cup"] \arrow[u, two heads, "\partial"'] & (F^2_c/F^1_c)H^3(Y,\mathbf{Q}_p(2)) \arrow[u, "\partial"', "\wr"] \arrow[ur, "\mathrm{Tr}_Y"', "\sim"] &
\end{tikzcd}$$

Map $\beta_{D,i}$ is induced by the top pairing in this diagram and we want to show that that this pairing is perfect. Map can is injective by Lemma 8.9 and maps ∂ are surjective by Lemma 8.12. Identifying the graded pieces, the above commutative diagram can be rewritten as:

$$\begin{tikzcd}
H^{i-1}(\mathscr{G}_K,\frac{\mathscr{O}(D_C)}{C}(j-1))\otimes^{\square}_{\mathbf{Q}_p}H^{2-i}(\mathscr{G}_K,\frac{\mathscr{O}(Y_C)}{\mathscr{O}(D_C)}(1-j)) \arrow[r, "\cup^{\text{proét}}"] \arrow[d, hook, "\mathrm{can}"] & \mathbf{Q}_p \\
H^{i-1}(\mathscr{G}_K,\frac{\mathscr{O}(Y_C)}{C}(j-1))\otimes^{\square}_{\mathbf{Q}_p}H^{2-i}(\mathscr{G}_K,\frac{\mathscr{O}(Y_C)}{C}(1-j)) \arrow[r, "\cup^{G-coh}"] \arrow[u, two heads, "\partial"'] \arrow[ur, "\cup^{\text{proét}}"'] & H^1(\mathscr{G}_K,C), \arrow[u, "\mathrm{Tr}'_K"']
\end{tikzcd}\tag{8.15}$$

where Tr'_K is the composition

$$H^1(\mathscr{G}_K,C)\xleftarrow[\sim]{\frac{\log\chi}{\log\chi(\gamma_K)}}K\xrightarrow{\mathrm{Tr}_{K/\mathbf{Q}_p}}\mathbf{Q}_p.$$

The right triangle commutes by Theorem 7.4. Going back to the definition of cohomology with compact support, it is easy to check that the vertical maps (can and ∂) are induced from the canonical coherent maps. Moreover, using the generalized Tate's isomorphisms (3.16), we can rewrite the nontrivial cases of the above commutative diagram further as:

$$\begin{tikzcd}
\frac{\mathscr{O}(D)}{K}\otimes^{\square}_{\mathbf{Q}_p}\frac{\mathscr{O}(Y)}{\mathscr{O}(D)} \arrow[r, "\cup^{\text{proét}}"] \arrow[d, hook, "\mathrm{can}"] & \mathbf{Q}_p \\
\frac{\mathscr{O}(Y)}{K}\otimes^{\square}_{\mathbf{Q}_p}\frac{\mathscr{O}(Y)}{K} \arrow[r, "\cup"] \arrow[u, two heads, "\partial"'] \arrow[ur, "\cup^{\mathrm{coh}}"'] & K. \arrow[u, "\mathrm{Tr}_K"']
\end{tikzcd}\tag{8.16}$$

It is now clear that the top product in this diagram is the coherent product. Since the latter is perfect, it follows that so is the top product in diagram (8.15), as wanted.

The argument for the map

$$\gamma^c_{D,i}:\quad H^i_c(D,\mathbf{Q}_p(j))\xrightarrow{\sim}H^{4-i}(D,\mathbf{Q}_p(2-j))^{\cdot}$$

is analogous. $\square$

8.4. **The case of an open annulus.** We will now prove Theorem 8.1 for an open annulus. Let A be an open annulus over K.

Proposition 8.17. (Arithmetic duality for an open annulus) *Theorem* (8.1) *holds for A.*

Proof. Let $Y:=\partial A$ be the boundary of A, a disjoint union of two ghost circles Y_a, Y_b over K.

(i) The geometric and arithmetic trace maps are defined as follows:

$$\begin{aligned}
&\mathrm{Tr}_{A_C}:H^2_c(A_C,\mathbf{Q}_p(1))\to\mathbf{Q}_p,\\
&\mathrm{Tr}_A:H^4_c(A,\mathbf{Q}_p(2))\xrightarrow{\sim}H^2(\mathscr{G}_K,H^2_c(A_C,\mathbf{Q}_p(2)))\xrightarrow{H^2(\mathscr{G}_K,\mathrm{Tr}_{A_C})}\mathbf{Q}_p,
\end{aligned}\tag{8.18}$$

where Tr_{A_C} is the map coming from (5.21). The map Tr_A is an isomorphism by Lemma 5.23 and the vanishing of $H^2(\mathscr{G}_K, \mathscr{O}(\partial A_C)/(\mathscr{O}(A_C)\oplus C)(1))$ (see (5.21)).

Alternatively, we have an exact sequence

$$0 \to H^1(A_C, \mathbf{Q}_p(1)) \to H^1(Y_{a,C}, \mathbf{Q}_p(1)) \oplus H^1(Y_{b,C}, \mathbf{Q}_p(1)) \xrightarrow{\partial} H^2_c(A_C, \mathbf{Q}_p(1)) \to 0. \tag{8.19}$$

The trace map Tr_{A_C} is induced from the trace map

$$\mathrm{Tr}_{Y_{a,C}} + \mathrm{Tr}_{Y_{b,C}} : H^1(Y_{a,C}, \mathbf{Q}_p(1)) \oplus H^1(Y_{b,C}, \mathbf{Q}_p(1)) \to \mathbf{Q}_p.$$

This works because $H^1(A_C, \mathbf{Q}_p(1)) \xrightarrow{\sim} \mathbf{Q}_p$ compatibly with the maps $\mathrm{Tr}_{Y_{a,C}}$ and $\mathrm{Tr}_{Y_{b,C}}$. Applying $H^2(\mathscr{G}_K, -)$ and Tr_K to the exact sequence (8.19), we obtain that the arithmetic trace Tr_A can be defined via the maps

$$\mathrm{Tr}_A : \ H^4_c(A, \mathbf{Q}_p(2)) \xleftarrow{\partial} H^3(Y_a, \mathbf{Q}_p(2)) \oplus H^3(Y_b, \mathbf{Q}_p(2)) \xrightarrow{\mathrm{Tr}_{Y_a}+\mathrm{Tr}_{Y_b}} \mathbf{Q}_p.$$

We used here the fact that the composition

$$H^3(A, \mathbf{Q}_p(2)) \to H^3(\partial A, \mathbf{Q}_p(2)) \xrightarrow{\mathrm{Tr}_{Y_a}+\mathrm{Tr}_{Y_b}} \mathbf{Q}_p$$

is 0.

(ii) We will first show that the pairings

$$H^i(A, \mathbf{Q}_p(j)) \otimes^{\square}_{\mathbf{Q}_p} H^{4-i}_c(A, \mathbf{Q}_p(2-j)) \xrightarrow{\cup} H^4_c(A, \mathbf{Q}_p(2)) \xrightarrow[\sim]{\mathrm{Tr}_A} \mathbf{Q}_p \tag{8.20}$$

induce isomorphisms

$$\gamma_{A,i} : \quad H^i(A, \mathbf{Q}_p(j)) \xrightarrow{\sim} H^{4-i}_c(A, \mathbf{Q}_p(2-j))^*.$$

(•) *Compatibility of pairings.* The pairing (8.20) is compatible with the pairing (7.2), i.e., the diagram

$$\begin{tikzcd} H^i(A, \mathbf{Q}_p(j)) \otimes^{\square}_{\mathbf{Q}_p} H^{i'}_c(A, \mathbf{Q}_p(j')) \arrow[r, "\cup"] \arrow[d, hook, "\mathrm{can}"] & H^{i+i'}_c(A, \mathbf{Q}_p(j+j')) \\ H^i(Y_a \sqcup Y_b, \mathbf{Q}_p(j)) \otimes^{\square}_{\mathbf{Q}_p} H^{i'-1}(Y_a \sqcup Y_b, \mathbf{Q}_p(j')) \arrow[r, "\cup"] \arrow[u, two heads, "\partial"'] & H^{i+i'-1}(Y_a \sqcup Y_b, \mathbf{Q}_p(j+j')) \arrow[u, two heads, "\partial"'] \end{tikzcd} \tag{8.21}$$

commutes (up to a sign). That is, for $a \in H^i(A, \mathbf{Q}_p(j))(S)$ and $b \in H^{i'-1}(Y_a \sqcup Y_b, \mathbf{Q}_p(j'))(S)$, where S is an extremally disconnected set, we have

$$\partial(\mathrm{can}(a) \cup b) = (-1)^i a \cup \partial(b).$$

This follows easily from the formulas in Section 7.3.2. The injectivity and surjectivity of the vertical maps in diagram (8.21) follows from the computations in Section 5.5.

(•) *Filtration on cohomology.* By Section 4.4 and Lemma 4.34, there exists an ascending filtration on $H^i(A, \mathbf{Q}_p(j))$:

$$F^2_{i,j} = H^i(A, \mathbf{Q}_p(j)) \supset F^1_{i,j} \supset F^1_{i,j} \supset F^{-1}_{i,j} = 0,$$

such that

$$F^2_{i,j}/F^1_{i,j} \simeq H^{i-1}(\mathscr{G}_K, \mathbf{Q}_p(j-1)), \quad F^1_{i,j}/F^0_{i,j} \simeq H^{i-1}(\mathscr{G}_K, \tfrac{\mathscr{O}(A_C)}{C}(j-1)),$$
$$F^0_{i,j}/F^{-1}_{i,j} \simeq H^i(\mathscr{G}_K, \mathbf{Q}_p(j)).$$

Lemma 8.22. *The canonical injection*

$$\mathrm{can} : H^i(A, \mathbf{Q}_p(j)) \hookrightarrow H^i(Y_a \sqcup Y_b, \mathbf{Q}_p(j))$$

is strict for the filtrations, i.e., the induced map

$$(F^{s+1}/F^s)H^i(A, \mathbf{Q}_p(j)) \to (F^{s+1}/F^s)H^i(Y_a \sqcup Y_b, \mathbf{Q}_p(j)), \quad s \geq -1, \tag{8.23}$$

is injective.

Proof. This is clear for $s=-1$ and $s\geq 1$. For $s=0$, the argument is analogous to the one used in the proof of Lemma 8.9. $\square$

($\bullet$) *Filtration on cohomology with compact support.* Similarly, there exists an ascending filtration on $H^i_c(A,\mathbf{Q}_p(j))$:

$$F^2_{c,i,j}=H^i_c(A,\mathbf{Q}_p(j))\supset F^1_{c,i,j}\supset F^0_{c,i,j}\supset F^{-1}_{c,i,j}=0,$$

such that

$$F^2_{c,i,j}/F^1_{c,i,j}\simeq H^{i-2}(\mathscr{G}_K,\mathbf{Q}_p(j-1)),\quad F^1_{c,i,j}/F^0_{c,i,j}\simeq H^{i-2}(\mathscr{G}_K,\tfrac{\mathscr{O}(\partial A_C)}{\mathscr{O}(A_C)\oplus C}(j-1)),$$
$$F^0_{c,i,j}/F^{-1}_{c,i,j}\simeq H^{i-1}(\mathscr{G}_K,H^1_c(A_C,\mathbf{Q}_p(j)))\simeq H^{i-1}(\mathscr{G}_K,\mathbf{Q}_p(j)).$$

We will visualize this filtration in the following way (to simplify the notation we removed the subscripts from cohomology):
(8.24)

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & F^0_{c,i,j}:=H^{i-1}(\mathscr{G}_K,H^1_c(A_C,\mathbf{Q}_p(j))) \arrow[r] \arrow[d, equal] & F^1_{c,i,j} \arrow[r] \arrow[d] & H^{i-2}(\mathscr{G}_K,\frac{\mathscr{O}(\partial A_C)}{\mathscr{O}(A_C)\oplus C}(j-1)) \arrow[r] \arrow[d] & 0\\
0 \arrow[r] & H^{i-1}(\mathscr{G}_K,H^1_c(A_C,\mathbf{Q}_p(j))) \arrow[r] & F^2_{c,i,j}:=H^i_c(A,\mathbf{Q}_p(j)) \arrow[r] \arrow[d] & H^{i-2}(\mathscr{G}_K,H^2_c(A_C,\mathbf{Q}_p(j))) \arrow[r] \arrow[d] & 0\\
 & & H^{i-2}(\mathscr{G}_K,\mathbf{Q}_p(j-1)) \arrow[r, equal] \arrow[d] & H^{i-2}(\mathscr{G}_K,\mathbf{Q}_p(j-1)) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

The diagram is a map of exact sequences with exact columns. The middle exact row comes from the filtration induced by the Hochschild-Serre spectral sequence (see Lemma 5.23). The right exact column is induced by the syntomic filtration from (5.21). The term $F^1_{i,j}$ is defined as the pullback of the top right square.

Lemma 8.25. *The map*

$$\partial: F^sH^{i-1}(\partial A,\mathbf{Q}_p(j))\to F^sH^i_c(A,\mathbf{Q}_p(j)),\quad s\geq -1,$$

is surjective.

Proof. This is clear for $s=-1$ and $s\geq 2$. For $s=0$, we need to check strict surjectivity of the canonical map

$$H^{i-1}(\mathscr{G}_K,H^0(\partial A_C,\mathbf{Q}_p(j)))\to H^{i-1}(\mathscr{G}_K,H^1_c(A_C,\mathbf{Q}_p(j))).$$

But, as follows from (5.21), the canonical map

$$H^0(\partial A_C,\mathbf{Q}_p(j))\to H^1_c(A_C,\mathbf{Q}_p(j))$$

is surjective with a Galois equivariant section.

For $s=1$, having done the case of $s=0$, it suffices to show that the map

$$\partial:\quad (F^2/F^1)H^{i-1}(\partial A,\mathbf{Q}_p(j))\to (F^2/F^1)H^i_c(A,\mathbf{Q}_p(j))$$

is surjective. This amounts to showing that the canonical map

$$H^{i-2}(\mathscr{G}_K,\tfrac{\mathscr{O}(\partial A_C)}{C}(j-1))\to H^{i-2}(\mathscr{G}_K,\tfrac{\mathscr{O}(\partial A_C)}{\mathscr{O}(A_C)\oplus C}(j-1))$$

is surjective. But, by formula (3.16) and its suitable analog, or both the domain and the target of this map are trivial or this map is isomorphic to the canonical map

$$\tfrac{\mathscr{O}(\partial A)}{K}\to\tfrac{\mathscr{O}(\partial A)}{\mathscr{O}(A)\oplus K}$$

whose strict surjectivity is clear. □

(•) *Pairings on the graded pieces.* From diagram (8.21), Lemma 8.25, and Theorem 7.4, it follows that the cup product pairing

$$\cup:\quad H^i(A,\mathbf{Q}_p(j))\otimes^{\square}_{\mathbf{Q}_p} H^{i'}_c(A,\mathbf{Q}_p(j'))\to H^{i+i'}_c(A,\mathbf{Q}_p(j+j'))$$

is compatible with the above filtrations. In particular, the subgroups

$$F^0_{i,j}=H^i(\mathscr{G}_K,\mathbf{Q}_p(j))\subset H^i(A,\mathbf{Q}_p(j)),$$
$$F^1_{c,4-i,2-j}\subset H^{4-i}_c(A,\mathbf{Q}_p(2-j))$$

are orthogonal and so are F^0_c and F^1. Hence we obtain the following commutative diagram with exact rows (all the vertical maps are induced from cup products and the trace map Tr_A)

(8.26)

$$\begin{tikzcd}
0 \arrow[r] & H^{i-1}(\mathscr{G}_K,\frac{\mathscr{O}(A_C)}{C}(j-1)) \arrow[r] \arrow[d, "\gamma_{A,i}"] & H^{i-1}(\mathscr{G}_K,H^1(A_C,\mathbf{Q}_p(j))) \arrow[r] \arrow[d, "\beta_{A,i}"] & H^{i-1}(\mathscr{G}_K,\mathbf{Q}_p(j-1)) \arrow[r] \arrow[d, "\alpha_{A,i}"] & 0\\
0 \arrow[r] & H^{2-i}(\mathscr{G}_K,\frac{\mathscr{O}(\partial A_C)}{\mathscr{O}(A_C)\oplus C}(1-j))^* \arrow[r] & (F^2_{c,4-i,2-j})^* \arrow[r] & H^{3-i}(\mathscr{G}_K,\mathbf{Q}_p(2-j))^* &
\end{tikzcd}$$

We claim that the maps $\alpha_{A,i}$ and $\gamma_{A,i}$ are isomorphisms. Using Theorem 7.4 and arguing as in the case of diagram (8.14) for the open disc, we easily check that the map $\alpha_{A,i}$ is induced by the Galois pairing; hence it is an isomorphism.

To identify map $\gamma_{A,i}$, consider the commutative diagram (we omitted the indices of filtrations and the subscripts of cohomology):

$$\begin{tikzcd}
(F^1/F^0)H^i(A,\mathbf{Q}_p(j))\otimes^{\square}_{\mathbf{Q}_p}(F^1_c/F^0_c)H^{4-i}_c(A,\mathbf{Q}_p(2-j)) \arrow[r, "\cup"] \arrow[d, hook, "\mathrm{can}"] & (F^2_c/F^1_c)H^4_c(A,\mathbf{Q}_p(2)) \arrow[r, "\mathrm{Tr}_A", "\sim"'] & \mathbf{Q}_p\\
(F^1/F^0)H^i(\partial A,\mathbf{Q}_p(j))\otimes^{\square}_{\mathbf{Q}_p}(F^1/F^0)H^{3-i}(\partial A,\mathbf{Q}_p(2-j)) \arrow[u, two heads, "\partial"] \arrow[r, "\cup"] & (F^2_c/F^1_c)H^3(\partial A,\mathbf{Q}_p(2)) \arrow[u, "\partial"] \arrow[r, "\sim", "\mathrm{Tr}_{\partial A}"'] & \mathbf{Q}_p^{\oplus 2} \arrow[u, "+"]
\end{tikzcd}$$

Map $\gamma_{A,i}$ is induced by the top pairing in this diagram and we want to show that that this pairing is perfect. Map can is injective by Lemma 8.22 and maps ∂ are surjective by Lemma 8.25. Identifying the graded pieces, the above commutative diagram can be rewritten as:

(8.27)

$$\begin{tikzcd}
H^{i-1}(\mathscr{G}_K,\frac{\mathscr{O}(A_C)}{C}(j-1))\otimes^{\square}_{\mathbf{Q}_p}H^{2-i}(\mathscr{G}_K,\frac{\mathscr{O}(\partial A_C)}{\mathscr{O}(A_C)\oplus C}(1-j)) \arrow[r, "\cup"] \arrow[d, hook, "\mathrm{can}"] & H^1(\mathscr{G}_K,C) \arrow[r, "\mathrm{Tr}_K"] & \mathbf{Q}_p\\
H^{i-1}(\mathscr{G}_K,\frac{\mathscr{O}(\partial A_C)}{C}(j-1))\otimes^{\square}_{\mathbf{Q}_p}H^{2-i}(\mathscr{G}_K,\frac{\mathscr{O}(\partial A_C)}{C}(1-j)) \arrow[u, two heads, "\partial"] \arrow[r, "\cup"] & H^1(\mathscr{G}_K,C)^{\oplus 2} \arrow[u, "+"] \arrow[r, "\oplus\mathrm{Tr}_K"] & \mathbf{Q}_p^{\oplus 2}, \arrow[u, "+"]
\end{tikzcd}$$

where, for now, the cup products are the ones induces from the pro-étale products. Going back to the definition of cohomology with compact support, it is easy to check that the vertical maps (can and ∂) are induced from the canonical coherent maps. Moreover, using the generalized Tate's isomorphisms (3.16), we can rewrite the nontrivial cases of the above commutative diagram further as:

(8.28)
$$\begin{tikzcd}
\frac{\mathscr{O}(A)}{K}\otimes^{\square}_{\mathbf{Q}_p}\frac{\mathscr{O}(\partial A)}{\mathscr{O}(A)\oplus K} \arrow[r, "\cup"] \arrow[d, hook, "\mathrm{can}"] & K \arrow[r, "\mathrm{Tr}_K"] & \mathbf{Q}_p\\
\frac{\mathscr{O}(\partial A)}{K}\otimes^{\square}_{\mathbf{Q}_p}\frac{\mathscr{O}(\partial A)}{K} \arrow[u, two heads, "\partial"] \arrow[r, "\cup"] & K^{\oplus 2} \arrow[u, "+"] \arrow[r, "\oplus\mathrm{Tr}_K"] & \mathbf{Q}_p^{\oplus 2}, \arrow[u, "+"]
\end{tikzcd}$$

where, again, the products are still the ones induced from the pro-étale products. Now, the identification of the bottom product follows from Theorem 7.4: it is just the coherent product. It is now clear that the top product in this diagram is also the coherent product. Since the latter is perfect, it follows that so is the top product in diagram (8.27), as wanted.

We have shown that we have an isomorphism

$$\beta_{A,i}: \quad (F^2/F^0)H^i(A,\mathbf{Q}_p(j)) \xrightarrow{\sim} (F^1H^{4-i}_c(A,\mathbf{Q}_p(2-j)))^*.$$

Similarly as above, we obtain the following commutative diagram with exact rows (again, all the vertical maps are induced from cup products and the trace map Tr_A)

$$\begin{tikzcd} 0 \arrow[r] & H^i(\mathscr{G}_K,\mathbf{Q}_p(j)) \arrow[r] \arrow[d, "\tilde{\alpha}_{A,i}"] & H^i(A,\mathbf{Q}_p(j)) \arrow[r] \arrow[d, "\tilde{\gamma}_{A,i}"] & H^{i-1}(\mathscr{G}_K, H^1(A_C,\mathbf{Q}_p(j))) \arrow[r] \arrow[d, "\beta_{A,i}", "\wr"'] & 0 \\ 0 \arrow[r] & H^{2-i}(\mathscr{G}_K,\mathbf{Q}_p(1-j))^* \arrow[r] & H^{4-i}_c(A,\mathbf{Q}_p(2-j))^* \arrow[r] & (F^2_{c,4-i,2-j})^* \arrow[r] & 0 \end{tikzcd} \tag{8.29}$$

Using Theorem 7.4, we easily check that the map $\tilde{\alpha}_{A,i}$ is induced by the Galois pairing; hence it is an isomorphism. It follows that so is the map $\tilde{\gamma}_{A,i}$, as wanted.

The arguments for the map

$$\gamma^c_{A,i}: \quad H^i_c(A,\mathbf{Q}_p(j)) \to H^{4-i}(A,\mathbf{Q}_p(2-j))^*$$

are analogous. □

8.5. **The case of wide open curves.** Now we pass to a special kind of Stein curves.

8.5.1. *Definition of wide opens.* A *wide open* (see [10, Sec. III] for a brief study) is a rigid analytic space isomorphic to the complement in a proper, geometrically connected, and smooth curve of finitely many closed discs. Examples of wide opens include open discs and annuli. A K-*wide open* is a rigid analytic space over K isomorphic to the complement in a proper, geometrically connected, and smooth curve over K of finitely many closed discs over K.

We will need the following fact:

Lemma 8.30. *Let X be a wide open over K. Then one can embed X into a proper, geometrically connected, and smooth curve $\overline{X}$ over K such that we have an admissible covering*

$$\overline{X} = X \cup_{i=1}^m \{D_i\},$$

where D_i's are disjoint discs over K with centers $\{x_i\}_{i=1}^m$, $x_i \in \overline{X}(K)$, such that the intersections $A_i := X \cap D_i$ are open annuli.

Proof. By definition, we can embed X into $\overline{X}$ as in the lemma with complementary disjoint closed discs $\overline{D}_j$. We embed these discs $\overline{D}_j$ into open discs D_j. By shrinking if necessary, we may insure that the open discs are disjoint as well. It is then clear that the intersections $A_j := X \cap D_j$ are open annuli, as wanted. □

8.5.2. *Theorem 8.1 for wide opens.*

Proposition 8.31. (Arithmetic duality for wide opens) *Let X be a wide open over K. Theorem 8.1 holds for X.*

Proof. By Lemma 8.30, X can be embedded into a proper smooth geometrically irreducible curve $\overline{X}$ over K. Using the same notation as in that lemma, we write D for the union of the open discs D_i's and A for the union of the open annuli A_i's (coming from the intersections of the D_i's and X).

For $j \in \mathbf{Z}$, we have the Mayer-Vietoris distinguished triangles:

$$\begin{gathered} \mathrm{R}\Gamma(\overline{X},\mathbf{Q}_p(j)) \to \mathrm{R}\Gamma(X,\mathbf{Q}_p(j)) \oplus \mathrm{R}\Gamma(D,\mathbf{Q}_p(j)) \to \mathrm{R}\Gamma(A,\mathbf{Q}_p(j)), \\ \mathrm{R}\Gamma_c(A,\mathbf{Q}_p(j)) \to \mathrm{R}\Gamma_c(X,\mathbf{Q}_p(j)) \oplus \mathrm{R}\Gamma_c(D,\mathbf{Q}_p(j)) \to \mathrm{R}\Gamma_c(\overline{X},\mathbf{Q}_p(j)). \end{gathered} \tag{8.32}$$

The first one comes from analytic descent of pro-étale cohomology and the second one from analytic co-descent of compactly supported pro-étale cohomology.

The following lemma will be needed later to pass from derived duality to classical duality. We will write $\mathbb{D}(-) := \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{Q}_p}(-, \mathbf{Q}_p)$ for the duality functor.

Lemma 8.33. *Let X be a geometrically connected smooth Stein curve over K. Let $j, s \in \mathbf{Z}$. We have a natural isomorphism*

$$H^j\mathbb{D}(\mathrm{R}\Gamma_c(X, \mathbf{Q}_p(s))) \simeq H_c^{-j}(X, \mathbf{Q}_p(s))^*,$$

Proof. We have the spectral sequence

$$E_2^{i,j} = \underline{\mathrm{Ext}}^i_{\mathbf{Q}_p}(H_c^{-j}(X, \mathbf{Q}_p(s)), \mathbf{Q}_p) \Rightarrow H^{i+j}\mathbb{D}(\mathrm{R}\Gamma_c(X, \mathbf{Q}_p(s))).$$

Hence, it suffices to show that

$$\underline{\mathrm{Ext}}^i_{\mathbf{Q}_p}(H_c^j(X, \mathbf{Q}_p(s)), \mathbf{Q}_p) \simeq \begin{cases} H_c^j(X, \mathbf{Q}_p(s))^* & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases} \tag{8.34}$$

Since, by Theorem 5.8, the solid $\mathbf{Q}_p$-vector space $V^j := H_c^j(X, \mathbf{Q}_p(s))$ is of compact type, it is an LS of compact type (by [29, Cor. 3.38]). That is, it can be written as a countable colimit of Smith spaces with injective trace class transition maps (see [29, Def. 3.34]): $V^j \simeq \mathrm{colim}_n V_n^{j,S}$, where $V_n^{j,S}$'s are Smith spaces. We compute

$$\begin{aligned} \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{Q}_p}(V^j, \mathbf{Q}_p) &\simeq \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{Q}_p}(\mathrm{colim}_n V_n^{j,S}, \mathbf{Q}_p) \simeq \mathrm{R}\lim_n \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{Q}_p}(V_n^{j,S}, \mathbf{Q}_p) \\ &\simeq \mathrm{R}\lim_n \underline{\mathrm{Hom}}_{\mathbf{Q}_p}(V_n^{j,S}, \mathbf{Q}_p) \simeq \lim_n \underline{\mathrm{Hom}}_{\mathbf{Q}_p}(V_n^{j,S}, \mathbf{Q}_p) \simeq \underline{\mathrm{Hom}}_{\mathbf{Q}_p}(\mathrm{colim}_n V_n^{j,S}, \mathbf{Q}_p) \\ &\simeq \underline{\mathrm{Hom}}_{\mathbf{Q}_p}(V^j, \mathbf{Q}_p). \end{aligned}$$

The third quasi-isomorphism follows from the fact that Smith spaces are projective objects, the fourth one from Section 2.2.4 since the pro-system $\{\underline{\mathrm{Hom}}_{\mathbf{Q}_p}(V_n^{j,S}, \mathbf{Q}_p)\}_{n\in\mathbf{N}}$ is built from Banach spaces with compact transition maps hence is equivalent to a pro-system of Banach spaces with dense transition maps (by [30, discussion after Prop. 16.5]). This proves (8.34), as wanted. $\square$

(i) *Duality map $\gamma_{X,i}$.* Apply now the duality functor $\mathbb{D}(-)$ to the second triangle in (8.32). We obtain a distinguished triangle

$$\mathbb{D}(\mathrm{R}\Gamma_c(\overline{X}, \mathbf{Q}_p(j))) \to \mathbb{D}(\mathrm{R}\Gamma_c(X, \mathbf{Q}_p(j))) \oplus \mathbb{D}(\mathrm{R}\Gamma_c(D, \mathbf{Q}_p(j))) \to \mathbb{D}(\mathrm{R}\Gamma_c(A, \mathbf{Q}_p(j))). \tag{8.35}$$

Here and below, we write $\mathrm{R}\Gamma(-) := \mathrm{R}\Gamma$, $\mathrm{R}\Gamma_c(-) := \mathrm{R}\Gamma_c(-)$. We have a map of distinguished triangles:

$$\begin{tikzcd} \mathrm{R}\Gamma(\overline{X}, \mathbf{Q}_p(j)) \arrow[r, "\gamma_{\overline{X}}"] \arrow[d] & \mathbb{D}(\mathrm{R}\Gamma_c(\overline{X}, \mathbf{Q}_p(2-j))[4]) \arrow[d] \\ \mathrm{R}\Gamma(X, \mathbf{Q}_p(j)) \oplus \mathrm{R}\Gamma(D, \mathbf{Q}_p(j)) \arrow[r, "\gamma_X \oplus \gamma_D"] \arrow[d] & \mathbb{D}(\mathrm{R}\Gamma_c(X, \mathbf{Q}_p(2-j))[4]) \oplus \mathbb{D}(\mathrm{R}\Gamma_c(D, \mathbf{Q}_p(2-j))[4]) \arrow[d] \\ \mathrm{R}\Gamma(A, \mathbf{Q}_p(j)) \arrow[r, "\gamma_A"] & \mathbb{D}(\mathrm{R}\Gamma_c(A, \mathbf{Q}_p(2-j))[4]) \end{tikzcd} \tag{8.36}$$

The top horizontal arrow is a quasi-isomorphism by Section 8.2. The bottom horizontal arrow is a quasi-isomorphism by Proposition 8.17 and Lemma 8.33. Since, moreover, by Proposition 8.5 and Lemma 8.33, the arrow γ_D in diagram (8.36) is a quasi-isomorphism, it follows that the map

$$\gamma_X : \mathrm{R}\Gamma(X, \mathbf{Q}_p(j)) \to \mathbb{D}(\mathrm{R}\Gamma_c(X, \mathbf{Q}_p(2-j))[4]), \tag{8.37}$$

is a quasi-isomorphism as well. That is, for $i \in \mathbf{N}$, we have an induced isomorphism

$$\gamma_{X,i} : H^i(X, \mathbf{Q}_p(j)) \xrightarrow{\sim} H^i\mathbb{D}(\mathrm{R}\Gamma_c(X, \mathbf{Q}_p(2-j))[4]). \tag{8.38}$$

This, in combination with Lemma 8.33, yields the isomorphism

$$\gamma_{X,i} : H^i(X, \mathbf{Q}_p(j)) \xrightarrow{\sim} H_c^{4-i}(X, \mathbf{Q}_p(2-j))^*,$$

as wanted.

(ii) *Duality map* $\gamma^c_{X,i}$. Write the map $\gamma^c_{X,i}$ as the composition

$$\gamma^c_{X,i}: H^i_c(X, \mathbf{Q}_p(j)) \xrightarrow[\sim]{\text{eval}} (H^i_c(X, \mathbf{Q}_p(j))^*)^* \xrightarrow[\sim]{\gamma^*_{X,4-i}} H^{4-i}(X, \mathbf{Q}_p(2-j))^*. \tag{8.39}$$

The evaluation map is an isomorphism since $H^i_c(X, \mathbf{Q}_p(j))$ is reflexive, hence LS, hence solid reflexive by [29, Thm. 3.40]. This proves that $\gamma^c_{X,i}$ is an isomorphism, as wanted. □

8.6. **The case of general Stein curves.** Finally, we are ready to treat general Stein curves.

Proposition 8.40. (Arithmetic duality for Stein curves) *Let X be a smooth geometrically irreducible Stein curve over K. Theorem 8.1 holds for X.*

Proof. (i) *Reduction step.* Take a Stein covering $\{X_n\}_{n\in\mathbf{N}}$ of X by dagger affinoids with adapted naive interiors X^0_n of X_n, $n \geq 1$. Using [10, proof of Prop. 3.3] we may choose X^0_n to be wide opens over finite extensions L_n of K. Since the duality map

$$\gamma_Y : \mathrm{R}\Gamma(Y, \mathbf{Q}_p(j)) \to \mathbb{D}(\mathrm{R}\Gamma_c(Y, \mathbf{Q}_p(2-j))[4])$$

satisfies étale descent, we know from Proposition 8.31 that it is a quasi-isomorphism for each X^0_n.

(ii) *Trace map.* We can write the trace map as

$$\mathrm{Tr}_X : \quad H^4_c(X, \mathbf{Q}_p(2)) \xleftarrow{\sim} \mathrm{colim}_n\, H^4_c(X^0_n, \mathbf{Q}_p(2)) \xrightarrow[\sim]{\mathrm{colim}_n \mathrm{Tr}_{X^0_n}} \mathbf{Q}_p.$$

(iii) *Duality map* $\gamma_{X,i}$. The duality map

$$\gamma_X : \mathrm{R}\Gamma(X, \mathbf{Q}_p(j)) \to \mathbb{D}(\mathrm{R}\Gamma_c(X, \mathbf{Q}_p(2-j))[4])$$

can be written as the composition (we set $s := 2-j$)

$$\begin{aligned}\mathrm{R}\Gamma(X, \mathbf{Q}_p(j)) &\xrightarrow{\sim} \mathrm{R}\lim_n \mathrm{R}\Gamma(X^0_n, \mathbf{Q}_p(j)) \xrightarrow[\sim]{\gamma_{X_n}} \mathrm{R}\lim_n \mathbb{D}(\mathrm{R}\Gamma_c(X^0_n, \mathbf{Q}_p(s))[4]) \\ &\simeq \mathbb{D}(\mathrm{colim}_n\, \mathrm{R}\Gamma_c(X^0_n, \mathbf{Q}_p(s))[4]) \xleftarrow{\sim} \mathbb{D}(\mathrm{R}\Gamma_c(X, \mathbf{Q}_p(s))[4]).\end{aligned}$$

The second quasi-isomorphism follows from (i).

On cohomology level this gives us isomorphisms

$$\gamma_{X,i} : H^i(X, \mathbf{Q}_p(j)) \xrightarrow{\sim} H^i\mathbb{D}(\mathrm{R}\Gamma_c(X, \mathbf{Q}_p(2-j))[4]).$$

By Theorem 5.8, $H^i_c(X, \mathbf{Q}_p(j))$ is of compact type hence $H^i\mathbb{D}(\mathrm{R}\Gamma_c(X, \mathbf{Q}_p(s))) \simeq (H^i_c(X, \mathbf{Q}_p(s)))^*$ by the computation above (proving (8.34)). Combination of these two observations shows that the duality map

$$\gamma_{X,i} : H^i(X, \mathbf{Q}_p(j)) \xrightarrow{\sim} H^{4-i}_c(X, \mathbf{Q}_p(2-j))^*$$

is an isomorphism.

(iv) *Duality map* γ^c_X. Analogous to the argument used in the proof of Proposition 8.31. □

8.7. **The case of dagger affinoid curves.** Finally, we will treat dagger affinoids of dimension 1.

Proposition 8.41. (Arithmetic duality for dagger affinoid curves) *Let X be a smooth geometrically irreducible dagger affinoid curve over K. Theorem 8.1 holds for X.*

Proof. Take a presentation $\{X_h\}_{h\in\mathbf{N}}$ of the dagger structure on X. Denote by X^0_h a naive interior of X_h adapted to $\{X_h\}$.

(i) *Duality map* $\gamma^c_{X,i}$. The duality map

$$\gamma^c_{X,i} : H^i_c(X, \mathbf{Q}_p(j)) \to H^{4-i}(X, \mathbf{Q}_p(2-j))^* \tag{8.42}$$

can be written as the composition

$$\begin{aligned}H^i_c(X, \mathbf{Q}_p(j)) &\xrightarrow{\sim} \lim_h H^i_c(X^0_h, \mathbf{Q}_p(j)) \xrightarrow[\sim]{\gamma^c_{X^0_h,i}} \lim_h H^{4-i}(X^0_h, \mathbf{Q}_p(2-j))^* \\ &\simeq (\mathrm{colim}_h\, H^{4-i}(X^0_h, \mathbf{Q}_p(2-j)))^* \simeq (H^{4-i}(X, \mathbf{Q}_p(2-j)))^*.\end{aligned}$$

The first isomorphism follows from the fact that $\mathrm{R}^1 \lim_h H^{i-1}_c(X^0_h, \mathbf{Q}_p(j)) = 0$ by Section 2.2.4 since the pro-system $\{H^{i-1}_c(X^0_h, \mathbf{Q}_p(j))\}_{h\in\mathbf{N}}$ is built from compact type spaces with compact transition maps (hence it is equivalent to a pro-system of Banach spaces with dense transition maps (by [30, by discussion after Prop. 16.5]). The second isomorphism is induced by the analog of the isomorphism (8.39). It follows that the duality map (8.42) is an isomorphism, as wanted.

(ii) *Duality map* $\gamma_{X,i}$. Write the duality map

$$\gamma_{X,i} : H^i(X, \mathbf{Q}_p(j)) \to H^{4-i}_c(X, \mathbf{Q}_p(2-j))^*$$

as the composition

$$\gamma_{X,i} : H^i(X, \mathbf{Q}_p(j)) \xrightarrow[\sim]{\text{eval}} (H^i(X, \mathbf{Q}_p(j))^*)^* \xrightarrow[\sim]{\gamma^{c,*}_{X,4-i}} H^{4-i}_c(X, \mathbf{Q}_p(2-j))^*.$$

The evaluation map is an isomorphism since $H^i(X, \mathbf{Q}_p(j))$ is reflexive by Proposition 4.23, hence LS, hence solid reflexive by [29, Thm. 3.30]. This proves that $\gamma_{X,i}$ is an isomorphism, as wanted. □

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