Syntomic complexes and p-adic nearby cycles

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(joint work with Pierre Colmez)

Let $\mathscr{O}_K$ be a complete discrete valuation ring with fraction field K of characteristic 0 and with perfect residue field k of characteristic p. Let $\mathscr{O}_F = W(k)$ and $F = \mathscr{O}_F[\frac{1}{p}]$ so that K is a totally ramified extension of F; let $e = [K : F]$ be the absolute ramification index of K. Let $\overline{\mathscr{O}}_K$ denote the integral closure of $\mathscr{O}_K$ in $\overline{K}$. Set $G_K = \mathrm{Gal}(\overline{K}/K)$, and let $\varphi = \varphi_{W(\overline{k})}$ be the absolute Frobenius on $W(\overline{k})$. For a log-scheme X over $\mathscr{O}_K$, X_n will denote its reduction mod p^n, X_0 will denote its special fiber.

0.1. **Statement of the main results.** Let X be a (strict) semistable scheme over $\mathscr{O}_K$. For $r \geq 0$, let $\mathscr{S}_n(r)_X$ be the (log) syntomic sheaf modulo p^n on $X_{0,\text{ét}}$. It can be thought of as a derived Frobenius and filtration eigenspace of crystalline cohomology or as a relative Fontaine functor. Fontaine-Messing [4] have defined a period map

$$\alpha^{\mathrm{FM}}_{r,n}: \quad \mathscr{S}_n(r)_X \to i^* Rj_* \mathbf{Z}/p^n(r)'_{X_{\mathrm{tr}}}$$

from syntomic cohomology to p-adic nearby cycles. Here $i : X_0 \hookrightarrow X$, $j : X_{\mathrm{tr}} \hookrightarrow X$, and X_{tr} is the locus where the log-structure is trivial. We set $\mathbf{Z}_p(r)' := \frac{1}{p^{a(r)}}\mathbf{Z}_p(r)$, for $r = (p-1)a(r) + b(r)$, $0 \leq b(r) \leq p-1$.

We prove that the Fontaine-Messing period map $\alpha^{\mathrm{FM}}_{r,n}$, after a suitable truncation, is essentially a quasi-isomorphism. More precisely, we prove the following theorem.

Theorem 0.1. *For $0 \leq i \leq r$, consider the period map*

$$\alpha^{\mathrm{FM}}_{r,n}: \quad \mathscr{H}^i(\mathscr{S}_n(r)_X) \to i^* R^i j_* \mathbf{Z}/p^n(r)'_{X_{\mathrm{tr}}}. \tag{0.2}$$

(i) *If K has enough roots of unity*[1] *then the kernel and cokernel of this map are annihilated by p^{Nr} for a universal constant N* (not depending on p, X, K, n or r).

(ii) *In general, the kernel and cokernel of this map are annihilated by p^N for an integer $N = N(e, p, r)$, which depends on e, r, but not on X or n.*

For $i \leq r \leq p-1$, it is known that a stronger statement is true: the period map

$$\alpha^{\mathrm{FM}}_{r,n}: \quad \mathscr{H}^i(\mathscr{S}_n(r)_X) \xrightarrow{\sim} i^* R^i j_* \mathbf{Z}/p^n(r)_{X_{\mathrm{tr}}}. \tag{0.3}$$

is an isomorphism for X a log-scheme log-smooth over a henselian discrete valuation ring $\mathscr{O}_K$ of mixed characteristic. This was proved by Kato [7], [8], Kurihara [10], and Tsuji [14], [15]. In [13] Tsuji generalized this result to some étale local

[1] The field F contains enough roots of unity and for any K, the field $K(\zeta_{p^n})$, for $n \geq c(K)+3$, where $c(K)$ is the conductor of K, contains enough roots of unity.

systems. As Geisser has shown [5], in the case without log-structure, the isomorphism (0.3) allows to approximate the (continuous) p-adic motivic cohomology (sheaves) of p-adic varieties by their syntomic cohomology; hence to relate p-adic algebraic cycles to differential forms.

As an application of Theorem 0.1, one can obtain the following generalization of the Bloch-Kato's exponential map [2]. Let $\mathscr{X}$ be a quasi-compact formal, semistable scheme over $\mathscr{O}_K$ (for example a semi-stable affinoid). For $i \geq 1$, consider the composition

$$\alpha_{r,i}: \quad H^{i-1}_{\mathrm{dR}}(\mathscr{X}_{K,\mathrm{tr}}) \to H^i(\mathscr{X},\mathscr{S}(r))_{\mathbf{Q}} \xrightarrow{\alpha_r^{FM}} H^i_{\text{ét}}(\mathscr{X}_{K,\mathrm{tr}},\mathbf{Q}_p(r)).$$

If X is a proper semistable scheme X over $\mathscr{O}_K$, and $1 \leq i \leq r-1$, then the G_K-representation $V_{i-1} = H^{i-1}_{\text{ét}}(X_{\overline{K}},\mathbf{Q}_p(r))$ is finite dimensional over $\mathbf{Q}_p$, $H^{i-1}_{\mathrm{dR}}(X_K)$ is finite dimensional over K, and $H^{i-1}_{\mathrm{dR}}(X_K) = D_{\mathrm{dR}}(V_{i-1})$. The map $\alpha_{r,i}$ for the formal scheme $\mathscr{X}$ associated to X is then the Bloch-Kato's map [11]

$$D_{\mathrm{dR}}(V_{i-1}) \to H^1(G_K, V_{i-1}) = H^i_{\text{ét}}(X_K,\mathbf{Q}_p(r)).$$

Easy comparison between de Rham and syntomic cohomologies, together with Theorem 0.1, yield the following result.

Corollary 0.4. *For $i \leq r-1$, the map*

$$\alpha_{r,i}: H^{i-1}_{\mathrm{dR}}(\mathscr{X}_{K,\mathrm{tr}}) \to H^i_{\text{ét}}(\mathscr{X}_{K,\mathrm{tr}},\mathbf{Q}_p(r))$$

is an isomorphism. Moreover, the map $\alpha_{r,r}: H^{r-1}_{\mathrm{dR}}(\mathscr{X}_{K,\mathrm{tr}}) \to H^r_{\text{ét}}(\mathscr{X}_{K,\mathrm{tr}},\mathbf{Q}_p(r))$ is injective (but not necessarily surjective: the case $i=r=1$ and $\mathscr{X} = \mathscr{O}_K^{\times}$ already provides a counterexample).

Recall how one shows that the period map $\alpha^{\mathrm{FM}}_{r,n}$ from (0.3) is an isomorphism. Under the stated assumptions one can do dévissage and reduce to $n=1$. Then one passes to the tamely ramified extension obtained by attaching the p'th root of unity ζ_p. There both sides of the period map (0.3) are invariant under twisting by $t \in \mathbf{A}_{\mathrm{cr}}$ and ζ_p, respectively, so one reduces to the case $r=i$. This is the Milnor case: both sides compute Milnor K-theory modulo p. To see this, one uses symbol maps from Milnor K-theory to the groups on both sides (differential on the syntomic side and Galois on the étale side). Via these maps all groups can be filtered compatibly in a way similar to the filtration of the unit group of a local field. Finally, the quotients can be computed explicitly by symbols [1], [6], [10], [13] and they turn out to be isomorphic. This approach to the computation of p-adic nearby cycles goes back to the work of Bloch-Kato [1] who treated the case of good reduction and whose approach was later generalized to semistable reduction by Hyodo [6].

Our proof is of very different nature: we construct another local (i.e., on affinoids of a special type, see below) period map, that we call $\alpha_r^{\mathscr{L}az}$. Modulo some

(φ, Γ)-modules theory reductions, it is a version of an integral Lazard isomorphism between Lie algebra cohomology and continuous group cohomology. We prove directly that it is a quasi-isomorphism and coincides with Fontaine-Messing's map up to constants as in Theorem 0.1. The (hidden) key input is the purity theorem of Faltings [3], Kedlaya-Liu [9], and Scholze [12]: it shows up in the computation of Galois cohomology in terms of (φ, Γ)-modules [9] which we use as a black box.

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